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A CATEGORY OF ARROW ALGEBRAS FOR MODIFIED REALIZABILITY

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CONTENTS

1

INTRODUCTION

> *Have you ever heard a category theorist say, "I want to prove that this diagram commutes:* let's suppose it doesn't"?
>
> — ANDREJ BAUER

This thesis aims to develop the theory of arrow algebras as a framework to study realizability toposes from a more concrete, 'algebraic', point of view which can also take localic toposes into account.

In this introduction, I will briefly present the main characters of this story, mostly sketching their development throughout the last century so as to have an overview of the field of research this thesis is concerned with. Since the mathematics of the following chapters will be written in a constructive metatheory, Section 1.1 starts by describing intuitionistic logic as an unformalized philosophy regarding mathematics. In Section 1.2, I will then introduce realizability, providing some formalization for the previous ideas. Some intuitive topos theory, and some motivation for the study thereof, is sketched in Section 1.3, which paves the way to Section 1.4 where the characters start to interact with each other. Finally, in Section 1.5, I will explain how this thesis is structured and what my contribution to the subject is.

1.1 INTUITIONISM

Intuitionism was originally developed by Luitzen Egbertus Jan Brouwer, at the beginning of the last century. Arising as a consequence of Brouwer's philosophy of mathematics, the heart of intuitionistic logic was in the rejection of the Law of Excluded Middle: that is, the principle according to which, for any statement ϕ, either ϕ or its negation $\neg\phi$ must be true. As the Law of Excluded Middle is equivalent to that of Double Negation Elimination, the principle according to which $\neg\neg\phi$ entails ϕ for any statement ϕ, to reject it means then to refuse proofs *by contradiction*: this reflects Brouwer's constructive ideology on mathematical entities, hence placing intuitionism in the more general realm of *constructive mathematics*.

As Brouwer notoriously abhorred any kind of formalism, the task of codifying a logical system for intuitionistic mathematics was carried out by Arend Heyting and Andrey Kolmogorov. The main challenge in developing formal intuitionistic logic lay in understanding how Brouwer conceived the logical symbols, and hence what it meant to give an intuitionistic proof of a statement. According to what would later come to be known as the *Brouwer-Heyting-Komogorov* (*BHK*) *interpretation* of the connectives, a proof of – for instance – an implication $\phi \to \psi$ consists of a procedure to convert proofs of ϕ into proofs of ψ. Clearly, this 'interpretation' still begged for an explanation itself, so as to formalize the primitive concepts of *proof* and *procedure*, fundamental in Brouwer's constructive vision of mathematics. For an account of constructive mathematics and intuitionism, we refer to the monograph [37], while [1] conversationally introduces the classical reader to what mathematics can look like constructively.

1.2 REALIZABILITY

In the seminal paper [22], in 1945, Stephen Cole Kleene started the field of realizability in the attempt to make explicit the algorithmic content of constructive proofs. To link intuitionism with the idea of 'effective computability', Kleene employed the theory of partial recursive functions which he himself had developed. The key idea was to associate to every sentence ϕ in the language of arithmetic a set of natural numbers which *realize* ϕ, in such a way that "n realizes ϕ" can be understood as "n provides evidence for the constructive truth of ϕ". Kleene's realizability offered a perfectly formal explanation of the BHK interpretation, hence allowing any classical mathematician to study intuitionistic logic.

If a sentence is provable from the axioms of arithmetic using intuitionistic logic[1], then it is realized. On the other hand, there exist sentences which are not provable, or even classically refutable, but which are realized: sentences of this kind are therefore consistent with constructive arithmetic, which means that they can be added without contradiction

1 That is, the system known as *Heyting arithmetic*, constructive analog of Peano arithmetic.

sometimes yielding counterintuitive results. A key example is given by Church's Thesis, stating that every function $\mathbb{N} \to \mathbb{N}$ is recursive:

$$\forall x \exists y \phi(x, y) \to \exists z \forall x (z \cdot x \downarrow \wedge \phi(x, z \cdot x))$$

where $z \cdot x$ represents the output of the z-th partial recursive function[2] on input x, and $z \cdot x\downarrow$ abbreviates "$z \cdot x$ is defined", both of which can be expressed within Heyting arithmetic itself. Over the years, realizability has therefore been used to prove the consistency of classically refutable principles, and to establish properties of formal systems for arithmetic, analysis and set theory, both intuitionistic and even classical: a historical survey of realizability in the twentieth century is given by [33].

The abstraction of the properties of $\mathbb{N}$ as the domain of realizers led Solomon Feferman to define *partial combinatory algebras*, introduced in [9], so as to 'do realizability' over more general structures. Partial combinatory algebras are abstract models of computation which capture the idea of a common domain for both the algorithms and the inputs/outputs to such algorithms: crucially, these algorithms are allowed to be partially defined, as it happens for partial recursive functions.

A variant of Kleene's number realizability which we will consider in this thesis, as the title suggests, is *modified realizability*, introduced by Georg Kreisel in [24]. The key idea behind modified realizability is to associate to every sentence a nonempty set of *potential realizers*, and a possibly empty subset of *actual realizers*. This version of realizability can be used to show the consistency of the Independence of Premise principle, that is:

$$(\neg\phi(y) \to \exists x \psi(x, y)) \to \exists x(\neg\phi(y) \to \psi(x, y))$$

or to show that Markov's principle, that is:

$$(\forall x(\phi(x) \vee \neg\phi(x)) \wedge \neg\neg\exists x \phi(x)) \to \exists x \phi(x)$$

is not provable within Heyting arithmetic.

1.3 TOPOS THEORY

Topos theory originated around 1960 from Alexander Grothendieck's work in algebraic geometry, which led to the notion of a *Grothendieck topos* as the category of sheaves on a *site*, generalizing the well-known notion

2 Assuming fixed an enumeration thereof.

of sheaves on a topological space. Then, around 1970, William Lawvere and Myles Tierney identified a number of elementary properties that Grothendieck toposes shared with the category of sets, which yielded the more general notion of an *elementary topos* – from now on, simply *topos*.

Toposes possess enough categorical structure to allow for an interpretation of typed higher-order logic, meaning that they can be regarded as alternate 'universes' in which to do mathematics instead of the *standard universe* of sets and functions. Indeed, many traditional constructions of set theory (such as powersets and sets of functions) can be carried out in every topos, and by reasoning internally we can prove facts about them as if we were talking about sets. Crucially, however, the reasoning must be carried out intuitionistically, as the internal logic of a topos need not be classical.[3] What this means is that classically impossible situations in 'our universe', the standard topos, may therefore be true elsewhere: for example, there exist toposes in which there are only countably many functions $\mathbb{N} \to \mathbb{N}$, or where every function $\mathbb{R} \to \mathbb{R}$ is continuous, or where $\mathbb{R}$ is uncountable (in the sense that there is no surjection $\mathbb{N} \to \mathbb{R}$) but there is an injection $\mathbb{R} \to \mathbb{N}$. Classical textbooks on topos theory are [26, 5, 19]; see [4], instead, for a survey focused on toposes as alternate mathematical universes.

1.4 REALIZABILITY TOPOSES

In the '70s, topos theory allowed logicians to encompass semantical ideas which were by then fully established, such as Cohen forcing for set theory or Kripke models for intuitionistic logic, under the concept of *Heyting-valued semantics*. Indeed, in 1973, Denis Higgs [12, 13] had proved how the category of H-valued sets, for a complete Heyting algebra H, is equivalent to the familiar topos of sheaves over H, which allows us to identify Heyting-valued semantics with the theory of localic toposes.

The construction of the topos of H-valued sets was split in [10] into two logically meaningful steps. First, H gives rise to a model of typed intuitionistic higher-order logic without equality, where types are sets and predicates of type X are functions $X \to H$, which are arranged in a Heyting algebra themselves with the pointwise order. Then, one formally 'adds equality' to the language in the form of an H-valued symmetric and

3 Besides, this is also often independent of whether classical logic is assumed to hold in the standard topos.

transitive relation for each type, which yields the topos of H-valued sets. In [16], Martin Hyland recognized that a similar two-step construction could be carried out considering the powerset of natural numbers $\mathcal{P}(\mathbb{N})$ in place of H, but with order and Heyting structure on sets of predicates governed by recursive function application. This resulted in the *effective topos* Eff, as it was named, which originated a whole new strand of research connecting realizability with topos theory. Indeed, it turned out that a sentence in the language of arithmetic is true in Eff if and only if it is realizable in Kleene's sense: in other words, the logic of the natural numbers inside the effective topos is exactly Kleene's realizability.

In [17], Martin Hyland, Peter Johnstone and Andrew Pitts showed how the first step of the construction could be greatly generalized, in particular in such a way as to recognize both the localic and the realizability examples as instances of the same construction. In doing so, they introduced the concept of a *tripos*, acronym for "topos-representing indexed partially-ordered set", which was the subject of Pitts' PhD thesis [35]. Many variations of realizability have been lifted to the framework of triposes and toposes, which allowed us to shed new light on their inter-relations and to study them systematically by studying the toposes which embodied them in their internal logic – that is, *realizability toposes*. Modified realizability, in particular, was first studied topos-theoretically by Robin Grayson in [11], who defined what is now called *Grayson's modified realizability topos* Mod; later references include [30] and [3].

Realizability toposes possess a number of peculiar properties also from a purely topos-theoretical perspective, starting with the fact that they constitute the prime example of elementary toposes which are *not* Grothendieck.[4] Realizability toposes can therefore be fruitfully exploited to find models for theories which classically do not have any, sometimes not even in any Grothendieck topos: over the years, this has found applications in areas such as synthetic domain theory, algebraic set theory and intuitionistic nonstandard arithmetic.

A great deal of research on realizability toposes has been carried out about functors between them, especially in relation with suitable notions of morphisms between partial combinatory algebras. A first such notion, known as *applicative morphisms*, was introduced in [25], then extended in [15] through the concept of *computational density*, which allowed us to

4 Of course, besides the much less interesting category of finite sets.

study *geometric morphisms* of realizability toposes from the point of view of the underlying partial combinatory algebras.

1.5 IN THIS THESIS

As said at the beginning, this thesis is concerned with the search for a concrete unifying framework to study toposes arising from triposes in a systematic way. Earlier work in this direction includes basic combinatorial objects [14], implicative algebras [28, 29] and evidenced frames [8]. *Arrow algebras* were introduced in [2] based on Marcus Briët's Master thesis [6], where it is shown how they are precisely the intermediate structure between partial combinatory algebras and the associated realizability triposes. In this thesis, we further this idea by defining appropriate categories of arrow algebras which perfectly factor through the construction of realizability triposes starting from partial combinatory algebras, but also encompassing the localic cases.

The outline of the thesis is as follows. In Chapter 2 and Chapter 3, we review the necessary background on triposes, toposes, and partial combinatory algebras which we will employ in later chapters. In Chapter 4, we describe arrow algebras as structures inducing triposes. The core of the thesis, as well as the main contribution to the field of research, is constituted by Chapter 5 and Chapter 6, where we introduce various notions of morphisms between arrow algebras and see how they correspond to morphisms between the associated triposes, in such a way as to recover both the localic and the realizability examples as particular instances. As an example of an application, in Chapter 7, we will employ the developed machinery to study modified realizability from the point of view of arrow algebras. Finally, Chapter 8, introduces the study of assemblies in the context of arrow algebras, generalizing the traditional notion of assemblies for partial combinatory algebras.

2 TRIPOSES AND TOPOSES

The notion of *tripos* was introduced in [17] and [35] as a categorical model of typed higher-order intuitionistic logic without equality, arising from the presentation of the topos of sheaves on a complete Heyting algebra H as the category of H-valued sets. Through the *tripos-to-topos* construction, every tripos gives rise to a topos in a process which essentially amounts to formally adding equality to the language. In this chapter, we review the necessary background on tripos theory, mainly following the account given in [34]; we will assume the reader to be familiar with topos theory.

2.1 PREORDER-ENRICHED CATEGORIES

First, we need to establish our terminology for 2-dimensional categories. Following [39], with *2-category* we mean a 2-dimensional category which is also an ordinary category, meaning that the unit and associativity laws for 1-cells hold on the nose. Instead, we speak of *bicategories* for 2-dimensional categories where the axioms of an ordinary category only hold up to (coherent) invertible 2-cells.

We can now introduce the most important kind of 2-dimensional category in the context of this thesis.

Definition 2.1. A *preorder-enriched* category is a locally small 2-category with at most one 2-cell between any pair of 1-cells.[1]

Explicitly, this means that a preorder-enriched category C is a category endowed with a preorder structure on each homset $C(A, B)$, in such a way that the composition map

$$C(B, C) \times C(A, B) \to C(A, C)$$

is order-preserving for all $A, B, C \in C$.

1 As in [39], we will also speak improperly of *preorder-enriched bicategories* for bicategories whose homcategories are preorders.

In a preorder-enriched category, a morphism $f : X \to Y$ is *left adjoint* to a morphism $g : Y \to X$ – equivalently, g is *right adjoint* to f – if $\mathrm{id}_X \leqslant gf$ and $fg \leqslant \mathrm{id}_Y$, in which case we write $f \dashv g$. Two parallel morphisms f, g are *isomorphic* if $f \leqslant g$ and $g \leqslant f$, in which case we write $f \simeq g$.

Example 2.2. Every category is preorder-enriched with respect to the discrete order.

Example 2.3. The category Preord of preordered sets and monotone functions is preorder-enriched with respect to the pointwise order.

Definition 2.4. Let C and D be preorder-enriched categories. A *pseudofunctor* $F : \mathsf{C} \to \mathsf{D}$ maps every object X in C to some object $F(X)$ in D and every morphism $f : X \to Y$ in C to some morphism $F(f) : F(X) \to F(Y)$ in D, in such a way that:

i. the association $\mathsf{C}(X, Y) \to \mathsf{D}(F(X), F(Y))$ is order-preserving for all objects X, Y in C;
ii. $F(\mathrm{id}_X) \simeq \mathrm{id}_{F(X)}$ for all objects X in C;
iii. $F(gf) \simeq F(g)F(f)$ for all composable arrows f, g in C.

In particular, F is a *2-functor*[2] if it is an actual functor, i.e. if the equalities in (ii) and (iii) hold on the nose rather than up to isomorphism.

Definition 2.5. Let F and G be pseudofunctors $\mathsf{C} \to \mathsf{D}$. A *pseudonatural transformation* $\Phi : F \Rightarrow G$ is given by a morphism $\Phi_X : F(X) \to G(X)$ in D for every object X in C in such a way that, for all morphisms $f : X \to Y$ in C, the square

$$\begin{array}{ccc} F(X) & \xrightarrow{\Phi_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\Phi_Y]{} & G(Y) \end{array}$$

commutes up to isomorphism.

Remark 2.6. We will also have to deal with *pseudomonads* on a preorder-enriched category C, that is, the datum of a pseudofunctor $T : \mathsf{C} \to \mathsf{C}$ and two pseudonatural transformations $\eta : \mathrm{id}_T \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ satisfying the usual monad laws up to (coherent) invertible modifications. We will not go into any detail about the theory of pseudomonads, which will serve us only *en passant*: for reference, see [18].

2 This is called an *enriched functor* in [34].

We will then consider *pseudoalgebras* over a pseudomonad T, that is, objects X endowed with a morphism $\mathsf{T}(\mathsf{X}) \to \mathsf{X}$ in C satisfying the usual algebra laws up to isomorphism. In complete analogy with the 1-dimensional case, pseudoalgebras determine the *Kleisli bicategory* $\mathsf{Kl}(\mathsf{T})$ of the pseudomonad T: the critical point here is precisely that $\mathsf{Kl}(\mathsf{T})$ is not necessarily a (2-)category, but we will see how this will not really be an issue for our purposes.

The following example of a preorder-enriched category plays a key role in the theory of triposes.

Definition 2.7. A *Heyting prealgebra* is a preorder whose poset reflection is a Heyting algebra; in other words, it is a (small) thin cartesian closed category which admits finite coproducts.

A morphism of Heyting prealgebras is a monotone function which is a morphism of Heyting algebras between the poset reflections of domain and codomain; in other words, it is a functor preserving finite products and coproducts and exponential objects.

We denote with HeytPre the category of Heyting prealgebras, which is preorder-enriched with respect to the pointwise order.

2.2 TRIPOSES

As the only triposes considered in this thesis will be Set-based, we restrict ourselves to triposes over a fixed elementary topos E.

TRIPOSES AND TRANSFORMATIONS OF TRIPOSES

Definition 2.8. An E-*tripos* is a pseudofunctor $\mathsf{P} : \mathsf{E}^{\mathsf{op}} \to \mathsf{HeytPre}$ satisfying the following axioms.

i. For every morphism $f : X \to Y$ in E, the map $f^* := \mathsf{P}(f) : \mathsf{P}(Y) \to \mathsf{P}(X)$ has both a left adjoint $\exists_f$ and a right adjoint $\forall_f$ in Preord.[3] Moreover, these adjoints satisfy the *Beck-Chevalley condition*, which means that

3 That is, $\exists_f$ and $\forall_f$ need not preserve the Heyting structure.

for every pullback square in E as the left one, the induced square on the right commutes up to isomorphism in Preord:[4]

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle h} \\ Z & \xrightarrow[k]{} & W \end{array} \qquad \begin{array}{ccc} P(X) & \xleftarrow{f^*} & P(Y) \\ {\scriptstyle \forall_g}\downarrow & & \downarrow{\scriptstyle \forall_h} \\ P(Z) & \xleftarrow[k^*]{} & P(W) \end{array}$$

ii. There exists a *generic element* in P, which is an element $\sigma \in P(\Sigma)$ for some object Σ in E with the property that, for every object X in E and every element $\phi \in P(X)$, there exists a morphism $[\phi] : X \to \Sigma$ in E such that ϕ and $[\phi]^*(\sigma)$ are isomorphic elements of $P(X)$.

An E-tripos P such that $P(X) = E(X, \Sigma)$ for some object Σ in E is said to be *canonically presented*.

Intuitively, one should think of E as the category of types and terms of a many-sorted higher-order intuitionistic language $\mathcal{L}$, and each $P(X)$ can be thought of a set of nonstandard *predicates* on X. The preorder on each $P(X)$ can then be interpreted as a *entailment* with respect to an $\mathcal{L}$-theory T, hence we denote it as $\vdash_X$; in particular, isomorphism in $P(X)$ is consequently denoted as $\dashv\vdash_X$.

Remark 2.9. The fact that f^* preserves the Heyting structure implies the *Frobenius condition*, that is, for every morphism $f : X \to Y$ in E, $\psi \in P(X)$ and $\phi \in P(Y)$:

$$\exists_f(\psi \wedge f^*(\phi)) \dashv\vdash_Y \exists_f(\psi) \wedge \phi$$

Example 2.10. A trivial example of an E-tripos is given by $\mathrm{Sub}_E : E^{op} \to$ HeytAlg. Indeed, for any morphism $f : X \to Y$ in E it is well-known that the pullback map $f^* : \mathrm{Sub}_E(Y) \to \mathrm{Sub}_E(X)$ is a morphism of Heyting algebras and has adjoints satisfying the Beck-Chevalley condition; a generic element is given by (the subobject of Ω represented by) the subobject classifier $t : 1 \to \Omega$.

Example 2.11. Let H be a complete Heyting algebra. We define the Set-tripos of H-valued predicates P_H as follows.

For any set X, we let $P_H(X) := \mathrm{Set}(X, H)$, which is a Heyting algebra under pointwise order and operations; for any function $f : X \to Y$, the precomposition map $f^* : P_H(Y) \to P_H(X)$ is then a morphism of Heyting

4 This also implies the same condition for left adjoints, namely that $\exists_f \circ g^* \simeq h^* \circ \exists_k$.

algebras. Adjoints for f^* are provided by completeness as, for $\phi \in P_H(X)$ and $y \in Y$:

$$\exists_f(\phi)(y) \coloneqq \bigvee_{x \in f^{-1}(y)} \phi(x) \qquad \forall_f(\phi)(y) \coloneqq \bigwedge_{x \in f^{-1}(y)} \phi(x)$$

which also satisfy the Beck-Chevalley condition. A generic element is trivially given by $\mathrm{id}_H \in P_H(H)$.

Definition 2.12. Let P and Q be E-triposes. A *transformation* $\Phi : P \to Q$ is a pseudonatural transformation $P \Rightarrow Q$ where P and Q are considered as pseudofunctors $E^{op} \to \mathsf{Preord}$; in other words, this means that each component $\Phi_X : P(X) \to Q(X)$ is an order-preserving map but not necessarily a morphism of Heyting prealgebras.

Transformations $P \Rightarrow Q$ can be ordered by letting $\Phi \leqslant \Psi$ if $\Phi_X \leqslant \Psi_X$ pointwise for all X in E, therefore making E-triposes and transformations into a preorder-enriched category which we denote as $\mathsf{Trip}(E)$.

A transformation $\Phi : P \to Q$ is an *equivalence* if there exists another transformation $\Psi : Q \to P$ such that $\Phi \circ \Psi \simeq \mathrm{id}_Q$ and $\Psi \circ \Phi \simeq \mathrm{id}_P$.

Remark 2.13. Through the generic element, every E-tripos is equivalent to a canonically presented one.

Remark 2.14. $\mathsf{Trip}(E)$ is essentially locally small: in fact, a transformation $\Phi : P \to Q$ is determined up to isomorphism by $\Phi_\Sigma(\sigma) \in Q(\Sigma)$, since for any $\phi \in P(X)$ we have that $\Phi_X(\phi) \dashv\vdash_X Q([\phi])(\Phi_\Sigma(\sigma))$.

INTERPRETATION OF LANGUAGES IN TRIPOSES Let P be an E-tripos.

Let $\mathcal{L}$ be the E-*typed language* for *higher order logic without equality*, that is, the language defined by:

- objects of E as types;
- morphisms $X_1 \times \cdots \times X_n \to Y$ in E as function symbols of type $(X_1, \ldots, X_n; Y)$;
- a fixed set of relational symbols, each with its type $(X_1, \ldots, X_n)$.

Given such a language, *terms* and *formulas* are defined inductively in the obvious way.

An *interpretation* of $\mathcal{L}$ in P is defined by assigning a predicate $[R] \in P(X_1 \times \cdots \times X_n)$ to every relation symbol of type $(X_1, \ldots, X_n)$. This assignment extends inductively[5] to all $\mathcal{L}$-formulas:

– the Heyting structure on each $P(X)$ and the maps f^* allow us to interpret the propositional connectives;

– the adjoints $\exists_\pi$, $\forall_\pi$ for suitable projections π allow us to interpret the quantifiers;

– the generic predicate and the cartesian closed structure of E allow us to interpret higher-order logic, where in particular Σ plays the role of the type of propositions.

With slight abuse of notation, we will typically identify a predicate $\phi \in P(X_1 \times \cdots \times X_n)$ with a relation symbol of type $(X_1, \ldots, X_n)$ interpreted as ϕ.

In particular, if ϕ is an $\mathcal{L}$-sentence, then $[\phi]$ is an element of $P(1)$: ϕ is *true* in P with respect to the interpretation $[\cdot]$ if $[\phi]$ is (isomorphic to) the top element in $P(1)$, in which case we write $P \vDash \phi$. These notions of interpretation and truth are sound with respect to a proof system for typed higher-order intuitionistic logic without equality: if ϕ is provable, then $P \vDash \phi$ for every E-tripos P and every interpretation $[\cdot]$ of $\mathcal{L}$ in P.

THE TRIPOS-TO-TOPOS CONSTRUCTION Let P be an E-tripos. We define a category $E[P]$ of *partial equivalence relations over* P as follows.

Objects of $E[P]$ are given by pairs $(X, \sim_X)$ where X is an object of E and $\sim_X$ is an *equality predicate*[6] for X, that is, an element of $P(X \times X)$ which is:

i. *symmetric*, $P \vDash \forall x, x'(x \sim_X x' \to x' \sim_X x)$;

ii. *transitive*, $P \vDash \forall x, x', x''(x \sim_X x' \wedge x' \sim_X x'' \to x \sim_X x'')$.

Morphisms $(X, \sim_X) \to (Y, \sim_Y)$ are isomorphism classes of *functional relations* from $(X, \sim_X)$ to $(Y, \sim_Y)$, i.e. predicates $F \in P(X \times Y)$ which are:

i. *strict*, $P \vDash \forall x, y(F(x, y) \to x \sim_X x \wedge y \sim_Y y)$;

ii. *relational*, $P \vDash \forall x, x', y, y'(F(x, y) \wedge x \sim_X x' \wedge y \sim_Y y' \to F(x', y'))$;

iii. *single-valued*, $P \vDash \forall x, y, y'(F(x, y) \wedge F(x, y') \to y \sim_Y y')$;

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5 The interested reader can find all the details in [34].

6 As we do not require reflexivity, i.e. $P \vDash \forall x(x \sim_X x)$, we can think of the statement $x \sim_X x$ as expressing that x *exists*.

iv. *total*, $P \vDash \forall x(x \sim_X x \to \exists y F(x,y))$.

The composition of a morphism $(X, \sim_X) \to (Y, \sim_Y)$ and a morphism $(Y, \sim_Y) \to (Z, \sim_Z)$ represented respectively by $F \in P(X \times Y)$ and $G \in P(Y \times Z)$ is the morphism $(X, \sim_X) \to (Z, \sim_Z)$ represented by the functional relation $[\exists y(F(x,y) \wedge G(y,z)] \in P(X \times Z)$, that is:

$$\exists_{\pi_{13}}(\pi_{12}^*(F) \wedge \pi_{23}^*(G))$$

where π_{12}, π_{13} and π_{23} are the projections from $X \times Y \times Z$. For every object $(X, \sim_X)$, the corresponding identity morphism is represented by $\sim_X \in P(X \times X)$ itself.

Remark 2.15. Two functional relations $F, G \in P(X \times Y)$ are isomorphic as soon as either $F \vdash_{X \times Y} G$ or $G \vdash_{X \times Y} F$ holds.

Theorem 2.16 (Pitts). $E[P]$ *is an elementary topos.*

Example 2.17. $E \simeq E[\mathrm{Sub}_E]$.

Example 2.18. For a complete Heyting algebra H, $\mathrm{Set}[P_H]$ is the topos of H-valued sets, equivalent to the topos $\mathrm{Sh}(H)$ of sheaves over H.

The advantage in presenting a topos as $E[P]$ for some E-tripos P lies in the fact that its internal logic, which is formulated *with* equality, can be reduced to the external logic of P – which was precisely missing the equality we added in the form of equality predicates. We briefly sketch here how this translation works for first-order logic; again, we refer to [34] for all the details.

Let $\mathcal{L}$ be a typed first-order language with equality and let $[\![\cdot]\!]$ be an interpretation of $\mathcal{L}$ in $E[P]$ in the usual sense of categorical logic, that is:

- each type σ is interpreted as an object $[\![\sigma]\!]$;
- each function symbol f of type $(\sigma_1, \dots, \sigma_n; \tau)$ is interpreted as a morphism $[\![f]\!] : [\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!] \to [\![\tau]\!]$;
- each relation symbol R of type $(\sigma_1, \dots, \sigma_n)$ is interpreted as a subobject $[\![R]\!]$ of $[\![\sigma_1]\!] \times \cdots \times [\![\sigma_n]\!]$, and in particular $=$ is interpreted as the diagonal subobject.

This assignment extends to $\mathcal{L}$-terms and $\mathcal{L}$-formulas in the usual way. The interpretation then reduces to the external logic of P as follows, making use of the fact that subobjects of $(X, \sim_X)$ correspond precisely to *strict relations* for $(X, \sim_X)$, i.e. predicates $R \in P(X)$ such that:

i. $P \vDash \forall x(R(x) \to x \sim_X x)$;
ii. $P \vDash \forall x, x'(R(x) \land x \sim_X x' \to R(x'))$.

For simplicity of exposition, we consider all terms and formulas to be in a common context, which can easily be obtained on the semantic side by composing with appropriate projections.

– Given two terms t, s whose interpretations are represented by functional relations $F, G \in P(X \times Y)$, then
$$[\![t = s]\!] = [\exists y(F(x, y) \land G(x, y))] \in P(X);$$
– given two formulas ϕ, ψ whose interpretations are subobjects of $(X, \sim_X)$ corresponding to strict relations $R, S \in P(X)$ for $(X, \sim_X)$, then
$$[\![\phi \land \psi]\!] = [R(x) \land S(x)] \qquad [\![\phi \lor \psi]\!] = [R(x) \lor S(x)],$$
whereas
$$[\![\phi \to \psi]\!] = [x \sim_X x \land (R(x) \to S(x))];$$
– given a formula ϕ whose interpretation is the subobject of $(Y, \sim_Y) \times (X, \sim_X)$ corresponding to the strict relation $R \in P(Y \times X)$, then
$$[\![\exists y \phi]\!] = [\exists y R(y, x)]$$
$$[\![\forall y \phi]\!] = [x \sim_X x \land \forall y(y \sim_Y y \to R(x, y))].$$

CONSTANT OBJECTS Let P be an E-tripos and let $E[P]$ be the associated topos.

We define the *constant object functor* $\nabla : E \to E[P]$ as follows:

– for every object X in E, $\nabla(X)$ is the object $(X, \exists_{\delta_X}(\top_X))$, where $\delta_X : X \to X \times X$ is the diagonal $\langle \mathrm{id}_X, \mathrm{id}_X \rangle$ and $\top_X$ is the top element in $P(X)$;
– for every morphism $f : X \to Y$ in E, $\nabla(f)$ is the morphism $\nabla(X) \to \nabla(Y)$ represented by the functional relation $\exists_{\langle \mathrm{id}_X, f \rangle}(\top_X) \in P(X \times Y)$.

The fact that ∇ is well-defined and functorial is proved in [35] and [34], but it also follows from the more general proofs carried out in Chapter 8. Moreover, note that ∇ preserves finite limits.

Example 2.19. The constant object functor for $E[\mathrm{Sub}_E]$ is one half of the equivalence $E \simeq E[\mathrm{Sub}_E]$.

Example 2.20. For a complete Heyting algebra H, $\nabla : \mathsf{Set} \to \mathsf{Set}[P_H]$ corresponds to the usual constant object functor $\Delta : \mathsf{Set} \to \mathsf{Sh}(H)$, inverse image of the unique geometric morphism $\mathsf{Sh}(H) \to \mathsf{Set}$.

The functor ∇ allows us to reconstruct P from the topos $\mathsf{E}[P]$, and also to characterize $\mathsf{E}[P]$ by means of a universal property.

Proposition 2.21. P *is equivalent to* $\mathrm{Sub}_{\mathsf{E}[P]}(\nabla(-))$.

Every regular category R admits an *ex/reg completion*, that is, an exact category $R_{ex/reg}$ together with a fully faithful regular functor $\eta : R \to R_{ex/reg}$ such that precomposition with η realizes an equivalence of categories:[7]

$$\mathsf{REG}(R, D) \simeq \mathsf{REG}(R_{ex/reg}, D)$$

for every exact category D. For more on ex/reg completions, we refer the reader to [27]. The details of the construction of $R_{ex/reg}$, together with the previous proposition, yield the following characterization.

Proposition 2.22. $\mathsf{E}[P]$ *is the ex/reg completion of its full subcategory on objects which embed into a constant object.*

2.3 GEOMETRIC MORPHISMS

The most important notion of morphism between toposes is arguably that of *geometric morphism*, which by now has a vast and standard theory. Much more niche, instead, is the theory of geometric morphisms of triposes, and how they relate with geometric morphisms of toposes: with no aim for a complete treatment, we review here the notions we will need in the following.

GEOMETRIC MORPHISMS OF TRIPOSES

Definition 2.23. Let P and Q be E-triposes. A transformation $\Phi : P \to Q$ is *left exact* if each component $\Phi_X : P(X) \to Q(X)$ preserves finite meets up to isomorphism. We denote with $\mathsf{Trip}_{\mathsf{lex}}(\mathsf{E})$ the wide subcategory of $\mathsf{Trip}(\mathsf{E})$ on left exact transformations.

A transformation $\Phi^+ : P \to Q$ admits a *right adjoint* if there exists another transformation $\Phi_+ : Q \to P$ such that $(\Phi^+)_X \dashv (\Phi_+)_X$ in Preord

7 We denote with $\mathsf{REG}(R, D)$ the category of regular functors $R \to D$ and natural transformations between them.

for all X in E, in which case we write $\Phi^+ \dashv \Phi_+$. If Φ^+ is moreover left exact, then the pair (Φ^+, Φ_+) defines a *geometric morphism* $Q \to P$[8], of which Φ_+ and Φ^+ constitute respectively the *direct* and *inverse image*. For practical reasons, we denote with $\mathsf{Trip}_{\mathsf{geo}}(\mathsf{E})$ the wide subcategory of $\mathsf{Trip}_{\mathsf{lex}}(\mathsf{E})$ on transformations having a right adjoint; a morphism $P \to Q$ in $\mathsf{Trip}_{\mathsf{geo}}(\mathsf{E})$ is hence a geometric morphism $Q \to P$.

Remark 2.24. Let $\Phi : P \to Q$ be an equivalence and let $\Psi : Q \to P$ be such that $\Phi \circ \Psi \simeq \mathrm{id}_Q$ and $\Psi \circ \Phi \simeq \mathrm{id}_P$. Then, $(\Phi, \Psi) : Q \to P$ and $(\Psi, \Phi) : P \to Q$ are both geometric morphisms.

Given a geometric morphism of toposes $(\Phi^+, \Phi_+) : \mathsf{E}[Q] \to \mathsf{E}[P]$ between toposes arising from E-triposes P and Q, we say that Φ_+ *preserves constant objects* if $\Phi^+ \circ \nabla_P \simeq \nabla_Q$.

Theorem 2.25. *Every geometric morphism of* E*-triposes* $Q \to P$ *induces a geometric morphism* $\mathsf{E}[Q] \to \mathsf{E}[P]$ *whose inverse image part preserves constant objects.*

Conversely, every geometric morphism $\mathsf{E}[Q] \to \mathsf{E}[P]$ *whose inverse image part preserves constant objects is induced by an essentially unique geometric morphism of* E*-triposes* $Q \to P$.

Example 2.26. Let X, Y be two complete Heyting algebras regarded as locales. Then, geometric morphisms $P_X \to P_Y$ correspond to locale homomorphisms $X \to Y$.

More precisely, for any geometric morphism $\Phi = (\Phi^+, \Phi_+) : P_X \to P_Y$ there exists an essentially unique morphism of locales $f : X \to Y$ such that, regarding f as a morphism of frames $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ and letting $f_* : \mathcal{O}(X) \to \mathcal{O}(Y)$ be its right adjoint, Φ^+ is given by postcomposition with f^* and Φ_+ is given by postcomposition with f_*.

SUBTRIPOSES

Definition 2.27. A geometric morphism of E-triposes $\Phi = (\Phi^+, \Phi_+) : Q \to P$ is an *inclusion* if either of the following equivalent conditions hold:

- for all X in E, $(\Phi_+)_X$ reflects the order;
- $\Phi^+ \circ \Phi_+ \simeq \mathrm{id}_Q$,

8 The direction is conventional and follows the same convention for geometric morphisms of toposes.

Dually, Φ is a *surjection* if either of the following equivalent conditions hold:

- for all X in E, $(\Phi^+)_X$ reflects the order;
- $\Phi_+ \circ \Phi^+ \simeq \mathrm{id}_P$.

Proposition 2.28. *Every geometric inclusion (resp. surjection) of* E*-triposes* $Q \to P$ *induces a geometric inclusion (resp. surjection)* $E[Q] \to E[P]$.

Moreover, every geometric inclusion into $E[P]$ *is induced, up to equivalence, by an essentially unique geometric inclusion of* E*-triposes into* P.

Definition 2.29. Let $\mathsf{SubTrip}(P)$ be the set of *subtriposes* of P, that is, triposes endowed with a geometric inclusions into P.[9] Given two geometric inclusions $\Phi : Q \hookrightarrow P$ and $\Psi : R \hookrightarrow P$, we write $\Phi \subseteq \Psi$ if there exists a geometric morphism $\Theta : Q \to R$ such that $\Phi \simeq \Psi \circ \Theta$ – meaning that $\Phi_+ \simeq \Psi_+ \circ \Theta_+$ or equivalently $\Phi^+ \simeq \Theta^+ \circ \Psi^+$ –, in which case Θ is an inclusion itself. This relation obviously makes $\mathsf{SubTrip}(P)$ into a preorder.

Two subtriposes $\Phi : Q \hookrightarrow P$ and $\Psi : R \hookrightarrow P$ are *equivalent* if they are isomorphic elements of $\mathsf{SubTrip}(P)$, that is, if both $\Phi \subseteq \Psi$ and $\Psi \subseteq \Phi$ hold; equivalently, this means that there exists an equivalence $\Theta : Q \to R$ such that $\Phi \simeq \Psi \circ \Theta$.

As it is known, subtoposes of a topos correspond up to equivalence to *local operators*, that is, morphisms $j : \Omega \to \Omega$ such that, in the internal logic of the topos:

i. $j(t) = t$;
ii. $jj = j$;
iii. $j(a \wedge b) = j(a) \wedge j(b)$,

In a topos of the form $E[P]$ for a canonically presented E-tripos $P := E(-, \Sigma)$, such a morphism corresponds to an essentially unique transformation $\Phi_j : P \to P$ which is:

i. left exact;
ii. *inflationary*, that is, $\mathrm{id}_P \leqslant \Phi_j$;
iii. *idempotent*, that is, $\Phi_j \Phi_j \simeq \Phi_j$.

Such Φ_j is called a *closure transformation* on P; conversely, every closure transformation on P determines a local operator on $E[P]$.

These correspondences lead to the following result.

9 For practical reasons, we identify a subtripos with the inclusion itself.

Theorem 2.30. *Let* P *be a canonically presented* E*-tripos and let* $\mathsf{ClTrans}(\mathsf{P})$ *be the set of closure transformations on* P*, ordered as above.*

1. *Geometric inclusions into* P *correspond, up to equivalence, to closure transformations on* P*; in particular, there is an equivalence of preorder categories:*

$$\mathsf{SubTrip}(\mathsf{P}) \simeq \mathsf{ClTrans}(\mathsf{P})^{op}$$

2. *Every geometric inclusion of toposes into* $\mathsf{E}[\mathsf{P}]$ *is, up to equivalence, of the form* $\mathsf{E}[\mathsf{Q}] \to \mathsf{E}[\mathsf{P}]$*, induced by a geometric inclusion of* E*-triposes* $\mathsf{Q} \to \mathsf{P}$*; in particular, there is an equivalence of preorder categories:*

$$\mathsf{SubTop}(\mathsf{E}[\mathsf{P}]) \simeq \mathsf{SubTrip}(\mathsf{P})$$

Remark 2.31. Note then that the poset reflection of $\mathsf{SubTrip}(\mathsf{P})$ is a bounded distributive lattice, since so is the set of subtoposes of any topos considered up to equivalence.

In the case of a canonically presented E-tripos $\mathsf{P} \coloneqq \mathsf{E}(-, \Sigma)$, we can even give an explicit description of the inclusion $\mathsf{Q} \to \mathsf{P}$ inducing a geometric inclusion into $\mathsf{E}[\mathsf{P}]$.

Let $(\mathsf{E}[\mathsf{P}])_j$ be the subtopos of $\mathsf{E}[\mathsf{P}]$ corresponding to a closure transformation $\Phi_j : \mathsf{P} \to \mathsf{P}$ and let $J \coloneqq (\Phi_j)_\Sigma(\mathrm{id}_\Sigma) : \Sigma \to \Sigma$. Then, $(\mathsf{E}[\mathsf{P}])_j$ is equivalent over E to $\mathsf{E}[\mathsf{P}_j]$, where P_j is the canonically presented E-tripos defined as follows:

- the underlying pseudofunctor is still $\mathsf{E}(-, \Sigma)$;
- the order $\vdash^j$ is redefined as $\phi \vdash^j_I \psi$ if and only if $\phi \vdash_I J\psi$;
- the implication $\to_j$ is redefined as

$$\Sigma \times \Sigma \xrightarrow{\mathrm{id}_\Sigma \times J} \Sigma \times \Sigma \xrightarrow{\ \to\ } \Sigma$$

  while $\top, \bot, \wedge, \vee$ remain unchanged.[10]

This means that we can restate the previous theorem as follows.

Corollary 2.32. *Let* P *be a canonically presented* E*-tripos.*

Then, every geometric inclusion of toposes into $\mathsf{E}[\mathsf{P}]$ *is induced, up to equivalence, by a geometric inclusion of triposes of the form:*

$$\mathsf{P}_j \underset{J\circ -}{\overset{\mathrm{id}_\Sigma \circ -}{\rightleftarrows}} \mathsf{P} \qquad (\bot)$$

10 Left and right adjoints for f^* can then be defined as $\phi \mapsto \exists_f(\phi)$ and $\phi \mapsto \forall_f(J\phi)$.

for some $J : \Sigma \to \Sigma$ *corresponding as above to a closure transformation* Φ_j *on* P.

Example 2.33. Let X be a complete Heyting algebra regarded as a locale. Then, closure transformations on P_X correspond to *nuclei* on X, that is, monotone, inflationary and idempotent endofunctions on the underlying frame of X; therefore, they also correspond to *sublocales* of X.

Another important notion from topos theory which can be recovered at the level of triposes is that of *open* and *closed* subtoposes.

Definition 2.34. A subtripos $\Phi : Q \hookrightarrow P$ is *open* if there exists an element $\alpha \in P(1)$ such that, for any $\phi \in P(I)$:

$$\Phi_+\Phi^+(\phi) \simeq P(!)(\alpha) \to \phi$$

where ! is the unique function $I \to 1$.

Dually, a geometric inclusion of triposes $\Psi : R \hookrightarrow P$ is *closed* if there exists an element $\beta \in P(1)$ such that, for any $\phi \in P(I)$:

$$\Psi_+\Psi^+(\phi) \simeq \phi \vee P(!)(\beta)$$

where ! is the unique function $I \to 1$.

For $\alpha = \beta$, Φ and Ψ define each other's complement in the lattice of subtriposes of P considered up to equivalence.

Corollary 2.35. *Through the correspondence in Theorem 2.30, open (resp. closed) subtriposes correspond to open (resp. closed) subtoposes.*

Example 2.36. Let X be a complete Heyting algebra regarded as a locale. Then, open (resp. closed) subtriposes of P_X correspond to *open* (resp. *closed*) *sublocales* of X.

3

PARTIAL COMBINATORY ALGEBRAS

Partial combinatory algebras[1] are the building blocks of realizability toposes. Our treatment of PCAs, as already in [2], closely follows that of [39], which allows us for the highest level of generality considered in the literature in what concerns appropriate notions of morphisms and their connections with morphisms of the associated toposes. We warn the reader that the following definitions and conventions are not entirely standard, but they turn out to be particularly convenient for our purposes. Moreover, we will only consider PCAs over Set as base category, as we will only develop the theory of arrow algebras over Set.

3.1 PARTIAL COMBINATORY ALGEBRAS

Definition 3.1. A *partial applicative poset* is a poset $(P, \leqslant)$ endowed with a partial binary map $\cdot : P \times P \to P$ called *application* such that if $a \leqslant a'$ and $b \leqslant b'$ and $a' \cdot b'$ is defined, then $a \cdot b$ is defined as well and $a \cdot b \leqslant a' \cdot b'$. $(P, \leqslant, \cdot)$ is *total* if $\cdot$ is a total operation, and it is *discrete* if $\leqslant$ is a discrete order.

We denote $a \cdot b$ also as ab, and we assume that $\cdot$ associates to the left. We write $ab\downarrow$ to indicate that ab is defined; note that a statement like "$abc\downarrow$" is to be interpreted as "$ab\downarrow$ and $(ab)c\downarrow$".

A *filter* on a partial applicative poset $(P, \leqslant, \cdot)$ is a subset $P^\# \subseteq P$ such that:

i. it is *upward closed*, i.e. if $a \in P^\#$ and $a \leqslant b$ then $b \in P^\#$;
ii. it is *closed under defined application*, i.e. if $a, b \in P^\#$ and $ab\downarrow$ then $ab \in P^\#$.

1 From now on, *PCAs*.

A *partial applicative structure* is the datum $(P, \leqslant, \cdot, P^\#)$ of a partial applicative poset $(P, \leqslant, \cdot)$ together with a filter $P^\#$ on it. In particular, it is *absolute* if $P^\# = P$.

A *partial combinatory algebra* is a partial applicative structure $(P, \leqslant, \cdot, P^\#)$ such that there exist elements $\mathbf{k}, \mathbf{s} \in P^\#$ satisfying:

i. $\mathbf{k}ab\downarrow$ and $\mathbf{k}ab \leqslant a$;

ii. $\mathbf{s}ab\downarrow$;

iii. if $ac(bc)\downarrow$, then $\mathbf{s}abc\downarrow$ and $\mathbf{s}abc \leqslant ac(bc)$,

for all $a, b, c \in P$. A partial combinatory algebra is *total* or *discrete* if the underlying partial applicative poset is, and it is *absolute* if it is absolute as a partial applicative structure.

Notation. Let $(P, \leqslant, \cdot)$ be a partial applicative poset. Given two possibly undefined expressions e, e' that, if defined, assume values in P, we write $e \preceq e'$ for the following statement: "if $e'\downarrow$, then $e\downarrow$ and $e \leqslant e'$".[2] Instead, we write $e \leqslant e'$ only if both expressions are always defined.

Example 3.2. The archetypal example of a PCA is *Kleene's first model* $\mathcal{K}_1$, underlying Kleene's original number realizability. $\mathcal{K}_1$ is the absolute and discrete PCA defined on $\mathbb{N}$ by letting $n \cdot m$ be the result of the n-th partial recursive function φ_n on input m whenever defined, fixed an ordering $\{ \varphi_i \mid i \in \mathbb{N} \}$ of the partial recursive functions $\mathbb{N} \to \mathbb{N}$.

Example 3.3. Underlying *function realizability*, a variant of Kleene's number realizability introduced in [23], is *Kleene's second model* $\mathcal{K}_2$, defined on the set $\mathbb{N}^\mathbb{N}$. To describe the application map, we need some notation.

Assume fixed a recursive coding of finite sequences of natural numbers where $[a_0, \ldots, a_{n-1}]$ codes the sequence $(a_0, \ldots, a_{n-1})$. For $\alpha \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$, we write $\alpha\big|_n$ for $[\alpha(0), \ldots, \alpha(n-1)]$ and $[n] * \alpha$ for the function $\alpha' \in \mathbb{N}^\mathbb{N}$ defined by $\alpha'(0) = n$ and $\alpha'(i+1) = \alpha(i)$. Let then $F_\alpha : \mathbb{N}^\mathbb{N} \rightharpoonup \mathbb{N}$ be the partial function defined by $F_\alpha(\beta) = m$ if and only if there exists an $n \in \mathbb{N}$ such that:

1. for all $i < n$, $\alpha(\beta\big|_i) = 0$;
2. $\alpha(\beta\big|_n) = m + 1$.

2 This relation is also called *Kleene inequality*.

With this notation, for $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$, we let $\alpha\beta\downarrow$ if $F_\alpha([n] \cdot \beta)\downarrow$ for every $n \in \mathbb{N}$, in which case $\alpha\beta$ is defined by $\alpha\beta(n) := F_\alpha([n] * \beta)$. $\mathcal{K}_2$ is then the discrete PCA obtained by choosing the set of total recursive functions as a filter.

The *van Oosten model* $\mathcal{B}$, introduced in [31], generalizes $\mathcal{K}_2$ by considering all *partial* functions $\mathbb{N} \rightharpoonup \mathbb{N}$ as its domain, and *partial* recursive functions as the filter; in that case, we obtain however a *total* PCA, which can be ordered by letting $\alpha \leqslant \beta$ if α extends β as partial functions $\mathbb{N} \rightharpoonup \mathbb{N}$ (in which case a filter is given by all subfunctions of partial recursive functions).

Elements of a PCA are usually called *combinators*; in particular, the **k** and **s** combinators correspond to constants from Schönfinkel's *combinatory logic*. Their most important consequence is *combinatory completeness*: every partial function obtained by repeatedly applying the application map is already present as a computation in the PCA itself.

More formally, let $\mathbb{P} = (P, \leqslant, \cdot, P^\#)$ be a PCA. The set of *terms* over $\mathbb{P}$ is defined recursively as follows:

- we assume given a countable set of distinct variables, each of which is a term;
- we assume given a constant symbol for each element in P, each of which is a term;
- if t_0, t_1 are terms, then so is $t_0 \cdot t_1$.

Every term $t = t(x_1, \dots, x_n)$ defines a partial function $P^n \rightharpoonup P$ which assigns the obvious, possibly undefined, interpretation $t(a_1, \dots, a_n)$ to an input sequence $(a_1, \dots, a_n) \in P^n$. Combinatory completeness can then be expressed as follows.

Proposition 3.4 (Combinatory completeness)**.** *For every nonempty sequence* $\underline{x}, y$ *of distinct variables and every term* $t = t(\underline{x}, y)$*, there exists an element* $\lambda^* \underline{x}, y.t \in P$ *such that, for every sequence* $\underline{a}$ *of the same length as* $\underline{x}$ *and every* b *in* P*:*

1. $(\lambda^* \underline{x}, y.t)\underline{a}\downarrow$;
2. $(\lambda^* \underline{x}, y.t)\underline{a}b \preceq t(\underline{a}, b)$.

Moreover, if all the constants occurring in t *are from* $P^\#$*, then* $\lambda^* \underline{x}, y.t \in P^\#$ *as well.*

Combinatory completeness allows us to perform constructions from recursion theory inside $\mathbb{P}$: let us see some examples which will turn out to be useful in the following. Since they also correspond to constants of combinatory logic, they are called combinators as well.

i. The *identity combinator* is defined as

$$\mathbf{i} \coloneqq \lambda^* x.x \in P^\#$$

and it satisfies $\mathbf{i}a \leqslant a$ for all $a \in P$.

ii. The *constant combinator* $\mathbf{k}$ can also be defined as $\lambda^* x.x$; its 'dual' $\overline{\mathbf{k}} \in P^\#$ is defined as

$$\overline{\mathbf{k}} \coloneqq \mathbf{ki}$$

and it satisfies $\overline{\mathbf{k}}ab \leqslant b$ for all $a, b \in P$.

iii. The *pairing combinator* $\mathbf{p} \in P^\#$ and the *unpairing combinators* $\mathbf{p}_0, \mathbf{p}_1 \in P^\#$ are defined as

$$\mathbf{p} \coloneqq \lambda^* x, y, z.zxy \qquad \mathbf{p}_0 \coloneqq \lambda^* x.x\mathbf{k} \qquad \mathbf{p}_1 \coloneqq \lambda^* x.x\overline{\mathbf{k}}$$

and they satisfy $\mathbf{p}_0(\mathbf{p}ab) \leqslant a$ and $\mathbf{p}_1(\mathbf{p}ab) \leqslant b$ for all $a, b \in P$.

The most important construction on an arbitrary PCA, at least in the context of this thesis, is the following.

Definition 3.5. Let $\mathbb{P} = (P, \leqslant, \cdot, P^\#)$ be a PCA.

The set TP of inhabited downward-closed subsets of P, ordered by inclusion, can be equipped with a partial applicative structure by defining, for $\alpha, \beta \in TP$:

$$\alpha \cdot \beta \coloneqq \downarrow\{\, xy \mid x \in \alpha,\ y \in \beta \,\}$$

in case $xy\downarrow$ for all $x \in \alpha$ and $y \in \beta$, and by defining the filter:

$$\begin{aligned}(TP)^\# &\coloneqq \left\{\, \alpha \in TP \;\middle|\; \alpha \cap P^\# \in TP \,\right\} \\ &= \left\{\, \alpha \in TP \;\middle|\; \exists \beta \in TP^\#,\ \beta \subseteq \alpha \,\right\} \\ &= \uparrow(TP^\#)\end{aligned}$$

Two combinators $\mathbf{k}, \mathbf{s} \in P^\#$ for $\mathbb{P}$ then yield corresponding combinators $\downarrow\{\mathbf{k}\}, \downarrow\{\mathbf{s}\} \in (TP)^\#$ for this partial applicative structure, making it into a PCA which we denote as $T\mathbb{P}$.

In the same way, the set DP of all downward-closed subsets of P ordered by inclusion can be made into a PCA $D\mathbb{P}$ with the same application operation, which in particular yields $\alpha\beta\downarrow$ and $\alpha\beta = \emptyset$ in case either $\alpha = \emptyset$

or $\beta = \emptyset$; a filter $(DP)^{\#}$ is given by the same $(TP)^{\#}$, and the combinators are described in the same way as well.

Remark 3.6. For future reference, given a discrete and absolute PCA $\mathbb{P}$, we denote with $\mathrm{Pow}(\mathbb{P})$ the PCA $D\mathbb{P}$: note then that the filter is given by the set of inhabited subsets of P.

Notation. Following [39], for $x \in P$ and $\beta \in DP$ we write $x \cdot \beta$ instead of $\downarrow\{x\} \cdot \beta$. Explicitly, $x\beta\downarrow$ if $xy\downarrow$ for all $y \in \beta$, in which case $x\beta = \downarrow\{\, xy \mid y \in \beta \,\}$.

Therefore, given $\gamma \in DP$, $x \cdot \beta \subseteq \gamma$ is equivalent to $xy \in \gamma$ for all $y \in \beta$.

3.2 MORPHISMS OF PARTIAL COMBINATORY ALGEBRAS

First, let us introduce a notion of morphism of PCAs which will not correspond exactly to functors between realizability toposes, but which we need for technical reasons in that perspective. Let $\mathbb{A} = (A, \leqslant, \cdot, A^{\#})$ and $\mathbb{B} = (B, \leqslant, \cdot, B^{\#})$ be PCAs.

Definition 3.7. A *morphism of PCAs* $\mathbb{A} \to \mathbb{B}$ is a function $f : A \to B$ satisfying:

i. $f(a) \in B^{\#}$ for all $a \in A^{\#}$;

ii. there exists an element $t \in B^{\#}$ such that if $aa'\downarrow$ then $tf(a)f(a')\downarrow$ and $tf(a)f(a') \leqslant f(aa')$;

iii. there exists an element $u \in B^{\#}$ such that if $a \leqslant a'$ then $uf(a)\downarrow$ and $uf(a) \leqslant f(a')$,

in which case we say that f is *realized* by $t, u \in B^{\#}$ or that it preserves application up to t and order up to u.

An order 'up to a realizer' can be defined on morphisms of PCAs as follows. Given two morphisms $f, f' : \mathbb{A} \to \mathbb{B}$, we write $f \leqslant f'$ if there exists some $s \in B^{\#}$ such that $sf(a)\downarrow$ and $sf(a) \leqslant f'(a)$ for all $a \in A$, in which case we say that $f \leqslant f'$ is *realized* by s.

Proposition 3.8. *PCAs, morphisms of PCAs and their order form a preorder-enriched category* OPCA.

Definition 3.9. A morphism of PCAs $f : \mathbb{A} \to \mathbb{B}$ is *computationally dense* if there exists an element $m \in B^{\#}$ with the property that for all $s \in B^{\#}$ there is some $r \in A^{\#}$ such that $mf(ra) \preceq sf(a)$ for all $a \in A$.

To move from these morphisms to the right ones, let us note that T and D defined in Definition 3.5 determine pseudomonads on OPCA.

Given a morphism of PCAs $f : \mathbb{A} \to \mathbb{B}$, we can define a morphism $\mathsf{T}f : \mathsf{T}\mathbb{A} \to \mathsf{T}\mathbb{B}$ by letting:

$$\begin{aligned} \mathsf{T}f(\alpha) &:= \downarrow(f(\alpha)) \\ &= \downarrow\{ f(x) \mid x \in \alpha \} \end{aligned}$$

and this makes the association $\mathbb{A} \mapsto \mathsf{T}\,\mathbb{A}$ pseudofunctorial. Then, a pseudomonad structure is defined considering:

- as unit δ, the pseudonatural transformation : $\mathrm{id}_{\mathsf{oPCA}} \Rightarrow \mathsf{T}$ of components $\delta_{\mathbb{A}} : \mathbb{A} \to \mathsf{T}\mathbb{A}$ given by the (computationally dense) morphisms of PCAs sending $a \in A$ to the *principal downset* $\downarrow\{a\} \in \mathsf{T}A$;
- as multiplication $\cup$, the pseudonatural transformation $\mathsf{TT} \Rightarrow \mathsf{T}$ of components $\cup_{\mathbb{A}} : \mathsf{TT}\,\mathbb{A} \to \mathsf{T}\,\mathbb{A}$ given by the (computationally dense) morphisms of PCAs sending $\alpha \in \mathsf{TT}A$ to its union $\bigcup \alpha$.[3]

Similarly, we have a (computationally dense) pseudomonad $(\mathsf{D}, \delta', \cup')$ on oPCA; the inclusions $\mathsf{T}A \hookrightarrow \mathsf{D}A$ determine a natural transformation $\mathsf{T} \Rightarrow \mathsf{D}$.

Through these two pseudomonads we can define two new notions of morphism of PCAs.

Definition 3.10. Let oPCA_T be the preorder-enriched bicategory defined as the Kleisli bicategory of the pseudomonad $(\mathsf{T}, \delta, \cup)$. Explicitly, OPCA_T is the category having PCAs as objects, and morphisms of PCAs $\mathbb{A} \to \mathsf{T}\,\mathbb{B}$ as morphisms $\mathbb{A} \to \mathbb{B}$, which we call *applicative morphisms*.

Similarly, let oPCA_D be the preorder-enriched bicategory defined as the Kleisli bicategory of the pseudomonad $(\mathsf{D}, \delta', \cup')$. Explicitly, OPCA_D is the category having PCAs as objects, and morphisms of PCAs $\mathbb{A} \to \mathsf{D}\,\mathbb{B}$ as morphisms $\mathbb{A} \to \mathbb{B}$, which we call *partial applicative morphisms*.

In either case, a morphism in OPCA_T or OPCA_D is computationally dense if it is so as a morphism of PCAs. Explicitly, a (partial) applicative morphism $f : \mathbb{A} \to \mathbb{B}$ is computationally dense if there exists an element $m \in B^\#$ with the property that for all $s \in B^\#$ there is some $r \in A^\#$ satisfying $mf(ra)\downarrow$ and $mf(ra) \subseteq sf(a)$ for any $a \in A$ such that $sf(a)\downarrow$.

Remark 3.11. As T and D are pseudomonads, OPCA_T and OPCA_D are not (preorder-enriched) categories but a priori only bicategories. The only

3 In this case, naturality actually holds on the nose.

axiom of an ordinary category that does not hold, however, is $f \circ \mathrm{id} = f$, in either $\mathsf{OPCA_T}$ and $\mathsf{OPCA_D}$. Since it can be shown that f satisfies $f \circ \mathrm{id} = f$ if and only if it preserves the order on the nose, f can therefore be replaced up to isomorphism with $f \circ \mathrm{id}$ which preserves the order on the nose: under this identification, $\mathsf{OPCA_T}$ and $\mathsf{OPCA_D}$ can then be treated as preorder-enriched categories.

Remark 3.12. An applicative morphism $f : \mathbb{A} \to \mathbb{B}$ can be seen as a partial applicative morphism $\mathbb{A} \to \mathbb{B}$ through $T\,\mathbb{B} \hookrightarrow D\,\mathbb{B}$. Note then that the inclusion $T\,\mathbb{B} \hookrightarrow D\,\mathbb{B}$ is a *pseudomono*, that is, the composition functor

$$\mathsf{OPCA}(\mathbb{A}, T\,\mathbb{B}) \to \mathsf{OPCA}(\mathbb{A}, D\,\mathbb{B})$$

is an equivalence of preorder categories for every PCA $\mathbb{A}$; therefore, $\mathsf{OPCA_T}$ is a preorder-enriched sub(-bi)category of $\mathsf{OPCA_D}$.

In other words, this means that we can reduce to consider only partial applicative morphisms in the following, of which 'plain' applicative morphisms are a particular case. More precisely, we can say that a partial applicative morphism $f : \mathbb{A} \to \mathbb{B}$ is *total* if $\mathrm{dom}\, f = A$, where

$$\mathrm{dom}\, f := \{\, a \in A \mid \exists b \in f(a) \,\},$$

which is equivalent to say that f factors through $T\,\mathbb{B} \hookrightarrow D\,\mathbb{B}$; hence, applicative morphisms can be identified with total partial applicative morphisms.

Example 3.13. The map $h(\alpha) := \{\, e \in \mathbb{N} \mid \varphi_e = \alpha \,\}$ is a partial applicative morphism $\mathcal{K}_2 \to \mathcal{K}_1$.

3.3 REALIZABILITY TRIPOSES

Let $\mathbb{A} = (A, \leqslant, \cdot, A^{\#})$ be a PCA. We define the *realizability tripos* $P_{\mathbb{A}}$ as follows.

For any set X, we let $P_{\mathbb{A}}(X) := \mathsf{Set}(X, DA)$, ordered by letting $\phi \vdash_X \psi$ if there exists an element $r \in A^{\#}$ such that $r \cdot \phi(x) \subseteq \psi(x)$[4] for all $x \in X$. The

4 Recall that this notation presupposes $r \cdot \phi(x)\downarrow$.

Heyting structure is then given by $\top_X(x) \coloneqq A$ and $\bot_X(x) \coloneqq \emptyset$ as top and bottom elements, and for $\phi, \psi \in P_{\mathbb{A}}(X)$:

$$(\phi \wedge \psi)(x) \coloneqq \mathbf{p} \cdot \phi(x) \cdot \psi(x)$$
$$(\phi \vee \psi)(x) \coloneqq (\mathbf{p} \cdot \mathbf{k} \cdot \phi(x)) \cup (\mathbf{p} \cdot \overline{\mathbf{k}} \cdot \psi(x))$$
$$(\phi \to \psi)(x) \coloneqq \{\, a \in A \mid a \cdot \phi(x) \subseteq \psi(x) \,\}$$

For any function $f : X \to Y$, the precomposition map $f^* : P_{\mathbb{A}}(Y) \to P_{\mathbb{A}}(X)$ is then a morphism of Heyting prealgebras. Adjoints for f^* satisfying the Beck-Chevalley condition are defined by:

$$\exists_f(\phi)(y) \coloneqq \bigcup_{x \in f^{-1}(y)} \phi(x)$$
$$\forall_f(\phi)(y) \coloneqq \left\{\, a \in A \;\middle|\; \forall b \in A,\ \forall x \in f^{-1}(y),\ ab{\downarrow} \text{ and } ab \in \phi(x) \,\right\}$$

while a generic element is trivially given by $\mathrm{id}_{DA} \in P_{\mathbb{A}}(DA)$.

We denote with $\mathsf{RT}(\mathbb{A})$ the *realizability topos* $\mathsf{Set}[P_{\mathbb{A}}]$.

Example 3.14. $\mathsf{RT}(\mathcal{K}_1)$ is the *effective topos* Eff.

3.4 TRANSFORMATIONS OF REALIZABILITY TRIPOSES

Let us now see how the notions of morphisms between PCAs defined above correspond to transformations between the associated realizability triposes and functors between the induced realizability toposes.

First, let's start by linking morphisms of PCAs with left exact transformations of realizability triposes. As we know, a transformation of triposes $\Phi : P_{\mathbb{A}} \to P_{\mathbb{B}}$ is given up to isomorphism by postcomposition with some function $f : DA \to DB$ at each component: as shown in [39, Prop. 3.3.16], Φ is left exact if and only if f is a morphism of PCAs, and the respective orders agree as well. In other words, we have the following.

Proposition 3.15. *The association* $f \mapsto f \circ -$ *is 2-functorial on downsets PCAs and, for any PCAs* $\mathbb{A}$ *and* $\mathbb{B}$*, it realizes an equivalence of preorder categories:*

$$\mathsf{OPCA}(D\,\mathbb{A}, D\,\mathbb{B}) \simeq \mathsf{Trip}_{\mathsf{lex}}(\mathsf{Set})(P_{\mathbb{A}}, P_{\mathbb{B}})$$

Instead, partial applicative morphisms $\mathbb{A} \to \mathbb{B}$ are characterized as those inducing *regular* transformations of triposes, which we now introduce.

Definition 3.16. Let P and Q be E-triposes. A left exact transformation $\Phi^+ : \mathsf{P} \to \mathsf{Q}$ is *regular* if it preserves existential quantification, that is, if:

$$(\Phi^+)_Y \circ \exists_g \dashv\vdash_Y \exists_g \circ (\Phi^+)_X$$

for all $g : X \to Y$ in E.

We denote with $\mathsf{Trip}_{\mathsf{reg}}(\mathsf{E})$ the wide subcategory of $\mathsf{Trip}_{\mathsf{lex}}(\mathsf{E})$ on regular transformations.

Remark 3.17. This means that Φ^+ preserves the interpretation of *regular logic*, the fragment of first-order logic defined by $\top$, $\wedge$ and $\exists$.

Consider now a partial applicative morphism $f : \mathbb{A} \to \mathbb{B}$, that is, a morphism of PCAs $f : \mathbb{A} \to \mathrm{D}\,\mathbb{B}$. Then, f corresponds to an essentially unique D-algebra morphism $\widetilde{f} : \mathrm{D}\,\mathbb{A} \to \mathrm{D}\,\mathbb{B}$ which, up to isomorphism, we can describe as:

$$\widetilde{f}(\alpha) \coloneqq \bigcup_{a \in \alpha} f(a)$$

The association $f \mapsto \widetilde{f}$ is 2-functorial[5] on $\mathsf{OPCA}_{\mathrm{D}}$, and it realizes an equivalence of preorder categories between partial applicative morphisms $\mathbb{A} \to \mathbb{B}$ and D-algebra morphisms $\mathrm{D}\,\mathbb{A} \to \mathrm{D}\,\mathbb{B}$.

Therefore, the correspondence stated in the previous proposition restricts to partial applicative morphisms and regular transformations: indeed, a left exact transformation $g \circ - : \mathsf{P}_{\mathbb{A}} \to \mathsf{P}_{\mathbb{B}}$ is regular if and only if $g : \mathrm{D}\,\mathbb{A} \to \mathrm{D}\,\mathbb{B}$ is a D-algebra morphism, i.e. if and only if it is up to isomorphism of the form $g = \widetilde{f}$ for an essentially unique partial applicative morphism $f : \mathbb{A} \to \mathbb{B}$.

Proposition 3.18. *The association* $f \mapsto \widetilde{f} \circ -$ *determines a 2-fully faithful 2-functor:*

$$\mathsf{OPCA}_{\mathrm{D}} \longrightarrow \mathsf{Trip}_{\mathsf{reg}}(\mathsf{Set})$$

Explicitly, this means that for any PCAs $\mathbb{A}$ *and* $\mathbb{B}$ *there is an equivalence of preorder categories:*

$$\mathsf{OPCA}_{\mathrm{D}}(\mathbb{A}, \mathbb{B}) \simeq \mathsf{Trip}_{\mathsf{reg}}(\mathsf{Set})(\mathsf{P}_{\mathbb{A}}, \mathsf{P}_{\mathbb{B}})$$

Finally, we can specify the previous correspondence to geometric morphisms by means of computational density. First, note how computational density can be characterized by the existence of right adjoints in OPCA.

5 As noted in Remark 3.11, $\mathsf{OPCA}_{\mathrm{D}}$ is only a bicategory, but compositions are defined on the nose so we can still speak of 2-functors rather than pseudofunctors.

Lemma 3.19. *Let* $f : \mathbb{A} \to \mathbb{B}$ *be a partial applicative morphism. Then, the following are equivalent:*

1. f *is computationally dense;*
2. $\widetilde{f} : D\,\mathbb{A} \to D\,\mathbb{B}$ *has a right adjoint in* OPCA,

in which case the right adjoint $h : D\,\mathbb{B} \to D\,\mathbb{A}$ *can be described as:*

$$h(\beta) := \downarrow\{\, a \mid mf(a)\downarrow \text{ and } mf(a) \subseteq \beta \,\}$$

where $m \in B^{\#}$ *witnesses computational density for* f.

As the existence of right adjoints in OPCA precisely corresponds to the existence of right adjoints on the level of transformations of triposes, we conclude with the following.

Theorem 3.20. *Let* $\mathsf{OPCA}_{D,cd}$ *be the wide sub(-bi)category of* OPCA_D *on computationally dense partial applicative morphisms.*

Then, the 2-functor of Proposition 3.18 restricts to a 2-fully faithful 2-functor:

$$\mathsf{OPCA}_{D,cd} \longrightarrow \mathsf{Trip}_{geo}(\mathsf{Set})$$

Explicitly, this means that for any PCAs $\mathbb{A}$ *and* $\mathbb{B}$ *there is an equivalence of preorder categories:*

$$\mathsf{OPCA}_{D,cd}(\mathbb{A}, \mathbb{B}) \simeq \mathsf{Trip}_{geo}(\mathsf{Set})(P_{\mathbb{A}}, P_{\mathbb{B}})$$

In particular, a right adjoint of $\widetilde{f} \circ - : P_{\mathbb{A}} \to P_{\mathbb{B}}$ *is given by* $h \circ - : P_{\mathbb{B}} \to P_{\mathbb{A}}$, *where* $h : D\,\mathbb{B} \to D\,\mathbb{A}$ *is right adjoint to* $\widetilde{f}$ *in* OPCA.

Example 3.21. Consider the function $f_0 : \mathcal{K}_1 \to \mathcal{K}_2$ defined by letting $f_0(n)$ be the constant function of value n. The function $f := \delta_{\mathcal{K}_2} f_0$ is a (total) applicative morphism $\mathcal{K} \to \mathcal{K}_2$ having the partial applicative morphism $h : \mathcal{K}_2 \to \mathcal{K}_1$ of Example 3.13 as right adjoint in OPCA. Therefore, the pair $f \dashv g$ induces a geometric morphism of triposes $P_{\mathcal{K}_2} \to P_{\mathcal{K}_1}$, and hence a geometric morphism of toposes $\mathsf{RT}(\mathcal{K}_2) \to \mathsf{Eff}$.

4

ARROW ALGEBRAS

We now introduce the main object of this thesis: *arrow algebras*. The material of this chapter, if not for minor remarks, is entirely taken from [2].

Arrow algebras were introduced in [2] generalizing Alexandre Miquel's *implicative algebras* [28, 29] as algebraic structures which induce triposes and hence toposes. The main advantage of arrow algebras, compared to implicative algebras, lies in the fact that they perfectly factor through the construction of realizability triposes coming from PCAs which are actually *partial*, whereas implicative algebras are the intermediate structure only in the case of *total* PCAs. Moreover, as we will see, subtriposes of triposes arising from arrow algebras – that is, *arrow triposes* – admit a particularly simple description as arrow triposes themselves, in a construction which does not work for implicative algebras. Together with the theory of morphisms between arrow algebras developed in the next chapters, this allows us to easily rephrase and extend results known in the literature at the level of PCAs: as an example of application, in Chapter 7, we will study modified realizability in the context of arrow algebras.

4.1 ARROW ALGEBRAS

Definition 4.1. An *arrow structure* is a complete meet-semilattice $(A, \preccurlyeq)$ endowed with a binary operation $\to : A \times A \to A$ such that if $a' \preccurlyeq a$ and $b \preccurlyeq b'$, then $a \to b \preccurlyeq a' \to b'$.

A *separator* on an arrow structure $(A, \preccurlyeq, \to)$ is a subset $S \subseteq A$ such that:

i. it is *upward closed*, i.e. if $a \in S$ and $a \preccurlyeq b$ then $b \in S$;

ii. it is *closed under modus ponens*, i.e. if $a \to b \in S$ and $a \in S$ then $b \in S$;

iii. it contains the *combinators*:

$$\mathbf{k} \coloneqq \curlywedge_{a,b \in A} a \to b \to a$$

$$\mathbf{s} \coloneqq \curlywedge_{a,b,c \in A} (a \to b \to c) \to (a \to b) \to (a \to c)$$

$$\mathbf{a} \coloneqq \curlywedge_{a,(b_i)_{i\in I},(c_i)_{i\in I} \in A} \left(\curlywedge_{i \in I} a \to b_i \to c_i \right) \to a \to \left(\curlywedge_{i \in I} b_i \to c_i \right)$$

An *arrow algebra* is a quadruple $(A, \preccurlyeq, \to, S)$ where $(A, \preccurlyeq, \to)$ is an arrow structure and S is a separator on it.

Notation. We will assume that $\to$ associates to the right, as it is common in type theory, and binds stronger than $\curlywedge$. This means that, for example, the combinator $\mathbf{k}$ is given by $\curlywedge_{a,b \in A}(a \to (b \to a))$.

A fundamental property of arrow algebras is given by the following.

Proposition 4.2. *Let $\mathcal{A} = (A, \preccurlyeq, \to, S)$ be an arrow algebra.*

1. *Let $(x_i)_{i\in I}, (y_i)_{i\in I}$ and $(z_i)_{i\in I}$ be I-indexed families of elements in A. If:*

$$\curlywedge_{i \in I} x_i \to y_i \to z_i \in S \quad \text{and} \quad \curlywedge_{i \in I} x_i \in S$$

   then:

$$\curlywedge_{i \in I} y_i \to z_i \in S$$

2. *Let ϕ be a propositional formula built from propositional variables $p_1, \ldots, p_n$ using implications only. If ϕ is an intuitionistic tautology, then:*

$$\curlywedge_{a_1,\ldots,a_n \in A} \phi(a_i/p_i) \in S$$

Remark 4.3. In the following, we will make extensive use of the previous proposition, which we will justify simply as *intuitionistic reasoning*.

The idea, in essence, is to find a suitable propositional intuitionistic tautology built of implications, in general of the shape $\phi \to \psi \to \chi$, so that $\curlywedge \phi \to \psi \to \chi \in S$ as in (2); then, from the knowledge of $\curlywedge \phi \in S$, we can deduce $\curlywedge \psi \to \chi \in S$ as in (1).

Corollary 4.4. *Let* $(A, \preccurlyeq, \to, S)$ *be an arrow algebra. Then,* S *contains the combinators:*

$$\mathbf{i} \coloneqq \bigcurlywedge_{a \in A} a \to a$$
$$\mathbf{b} \coloneqq \bigcurlywedge_{a,b,c \in A} (b \to c) \to (a \to b) \to (a \to c)$$

Definition 4.5. An arrow algebra $\mathcal{A} = (A, \preccurlyeq, \to, S)$ is *compatible with joins* if, for all $a \in A$ and all $B \subseteq A$:

$$(\bigcurlyvee_{b \in B} b) \to a = \bigcurlywedge_{b \in B} (b \to a)$$

NUCLEI Let us now introduce *nuclei*, a generalization of the locale-theoretical notion which will correspond to closure transformations on arrow triposes.

Definition 4.6. Let $\mathcal{A} = (A, \preccurlyeq, \to, S)$ be an arrow algebra. A *nucleus* on $\mathcal{A}$ is a function $j : A \to A$ such that:[1]

i. if $a \preccurlyeq b$ then $ja \preccurlyeq jb$;

ii. $\bigcurlywedge_{a \in A} a \to ja \in S$;

iii. $\bigcurlywedge_{a,b \in A} (a \to jb) \to ja \to jb \in S$.

In particular, these properties also imply:

iv. $\bigcurlywedge_{a \in A} jja \to ja \in S$;

v. $\bigcurlywedge_{a,b \in A} (a \to b) \to ja \to jb \in S$;

vi. $\bigcurlywedge_{a,b \in A} j(a \to b) \to ja \to jb \in S$,

and we can even substitute (iii) above with the conjunction of (iv) and (vi).

Drawing from the discussion in Chapter 2, we can define a new arrow algebra $\mathcal{A}_j$ starting from every nucleus j on $\mathcal{A}$. In hindsight, $\mathcal{A}_j$ will give rise precisely to the explicit description of the inclusion induced by the closure transformation corresponding to j given in Corollary 2.32.

---

1 Note how we systematically omit brackets to improve readability.

Proposition 4.7. *Let* $\mathcal{A} = (A, \preccurlyeq, \to, S)$ *be an arrow algebra and let* $j : A \to A$ *be a nucleus on it. Then,* $\mathcal{A}_j = (A, \preccurlyeq, \to_j, S_j)$ *with*

$$a \to_j b \coloneqq a \to jb \qquad S_j \coloneqq \{\, a \in A \mid ja \in S \,\}$$

is also an arrow algebra, which is compatible with joins whenever so is $\mathcal{A}$.

Remark 4.8. $\curlywedge_{a \in A} a \to ja \in S$ implies that $S \subseteq S_j$: indeed, if $a \in S$, then by modus ponens $ja \in S$, which precisely means $a \in S_j$.

Example 4.9. For any arrow algebra $\mathcal{A} = (A, \preccurlyeq, \to, S)$ and all $c \in A$, the following are nuclei on $\mathcal{A}$:

1. $ja \coloneqq c \to a$;
2. $ja \coloneqq (a \to c) \to c$;
3. $ja \coloneqq (a \to c) \to a$.

Particularly relevant in the theory of arrow algebras is the nucleus:

$$\partial a \coloneqq \top \to a$$

which is the special case of (1) above for $c = \top$. Note that it satisfies the extra property:

$$\bigcurlywedge_{a \in A} (\top \to a) \to a \in S$$

This follows by intuitionistic reasoning since $q \to (q \to p) \to p$ is an intuitionistic tautology and $\top \in S$.

The special case of (2) for $c = \bot$, instead, is the *double negation* nucleus:

$$\neg\neg a \coloneqq (a \to \bot) \to \bot$$

THE INTERPRETATION OF λ-TERMS Generalizing what happens for implicative algebras, let us now see how to interpret λ-terms in an arrow algebra $\mathcal{A} = (A, \preccurlyeq, \to, S)$.

First, let us recall the definition of λ-terms.

Definition 4.10. Fixed a countable set of *variables*, the collection of λ-*terms* is the smallest collection of formal expressions such that:

i. each variable is a λ-term;
ii. if M and N are λ-terms, then the *application* MN is a λ-term;

iii. if M is a λ-term and x is a variable, then the *abstraction* $\lambda x.M$ is a λ-term.

In order to interpret λ-terms in $\mathcal{A}$, we hence need to be able to interpret both applications and abstractions.

Definition 4.11. For $a, b \in A$, we define the *application*:

$$ab \coloneqq \curlywedge U_{a,b}$$

where $U_{a,b} \coloneqq \{ c \to d \in A \mid a \preccurlyeq b \to c \to d \}$.

Lemma 4.12. *The application enjoys the following properties.*

1. *If* $a \preccurlyeq a'$ *and* $b \preccurlyeq b'$, *then* $ab \preccurlyeq a'b'$.
2. $(a \to b \to c)a \preccurlyeq b \to c$.
3. $(a \to \curlywedge_{i \in I} b_i \to c_i)a \preccurlyeq \curlywedge_{i \in I} b_i \to c_i$.
4. *If* $a, b \in S$, *then* $ab \in S$.

Definition 4.13. For a function $f : A \to A$, we define the *abstraction*:

$$\lambda f \coloneqq \curlywedge_{x \in A} x \to \partial f(x)$$

Lemma 4.14. *Let* $f, g : A \to A$ *be functions; the abstraction enjoys the following properties.*

1. *If* $f(a) \preccurlyeq g(a)$ *for all* $a \in A$, *then* $\lambda f \preccurlyeq \lambda g$.
2. $(\lambda f)a \preccurlyeq \partial f(a)$ *for all* $a \in A$.

Notation. We will assume that application associates to the left and binds stronger than $\to$, which binds stronger than ∂, which binds stronger than $\curlywedge$. In particular, this means that $a \to bc$ stands for $a \to (bc)$, ∂ab stands for $\partial(ab)$, and $\partial a \to b$ stands for $\top \to a \to b$ rather than $(\top \to a) \to b$.

We can now recursively define an interpretation of λ-terms in $\mathcal{A}$.

Definition 4.15. Let M be a λ-term with free variables among $x_1, \ldots, x_n$. The *interpretation* ot M in $\mathcal{A}$ is a function $M^{\mathcal{A}} : A^n \to A$ defined recursively as follows:

1. if $M = x_i$, then $M^{\mathcal{A}}$ is the projection onto the i-th coordinate;

2. if $M = N_1 N_2$, then:

$$M^{\mathcal{A}}(\underline{a}) \coloneqq N_1^{\mathcal{A}}(\underline{a}) N_2^{\mathcal{A}}(\underline{a})$$

3. if $M = \lambda x.N$ where the free variables of N are among $x_1, \dots, x_n, x$, then:

$$M^{\mathcal{A}}(\underline{a}) \coloneqq \lambda(b \mapsto N^{\mathcal{A}}(\underline{a}, b))$$

The proof of the following nontrivial result needs a detour which is carried out in full details in [2].

Theorem 4.16. *Let $\mathcal{A} = (A, \preccurlyeq, \to, S)$ be an arrow algebra and let M be a λ-term with free variables among $x_1, \dots, x_n$. Then, for any $a_1, \dots, a_n \in S$:*

$$M^{\mathcal{A}}(a_1, \dots, a_n) \in S$$

Remark 4.17. The difference in the interpretation of λ-terms is arguably the biggest price to pay in moving to arrow algebras from implicative algebras, where it is much cleaner and better-behaved.

THE LOGICAL ORDER Although arrow algebras are defined in terms of $\preccurlyeq$, which we refer to as the *evidential order*, there is another, arguably more important, order which can be defined in terms of implications and separators. As we will see, this order will be the only relevant one in the construction of the arrow tripos.

Definition 4.18. Let $\mathcal{A} = (A, \preccurlyeq, \to, S)$ be an arrow algebra. We define the *logical order* $\vdash$ on A by letting:

$$a \vdash b \iff a \to b \in S$$

which is weaker than the evidential order since $\mathbf{i} \in S$ gives $a \to a \in S$ for all $a \in A$ and therefore if $a \preccurlyeq b$ then $a \to a \preccurlyeq a \to b \in S$ i.e. $a \vdash b$.

Note that $\vdash$ allows us to recover the separator[2] as:

$$S = \{ a \in A \mid \top \vdash a \}$$

Indeed, if $a \in A$ is such that $\top \vdash a$, then $a \in S$ follows by modus ponens from $\top, \top \to a \in S$; conversely, if $a \in S$, then the intuitionistic tautology $p \to q \to p$ gives $a \to \top \to a \in S$ and hence $\top \to a \in S$ by modus ponens.

2 In hindsight, this characterizes the separator as what Pitts calls the set of *designated truth values* of the induced arrow tripos.

Remark 4.19. We denote with $\vdash^j$ the logical order in the arrow algebra $\mathcal{A}_j$ induced by a nucleus j on $\mathcal{A}$. Explicitly, $a \vdash^j b$ if and only if $j(a \to jb) \in S$, which by the properties of nuclei and separators is equivalent to $a \to jb \in S$ and hence to $a \vdash jb$.

Proposition 4.20. *$(A, \vdash)$ is a Heyting prealgebra, with $\to$ being the Heyting implication.*

Proof (sketch). The combinators $\mathbf{i}, \mathbf{b} \in S$ respectively express reflexivity and transitivity of $\vdash$, making $(A, \vdash)$ into a preorder, which is bounded by $\top$ and $\bot$ since $\bot \preccurlyeq a \preccurlyeq \top$ for all $a \in A$. For $a, b \in A$, a meet of a and b can be defined as:

$$\begin{aligned} a \times b &:= (\lambda z.z(\partial a)(\partial b))^{\mathcal{A}} \\ &= \bigcurlywedge_{z \in A} z \to \partial z(\partial a)(\partial b) \end{aligned}$$

while a join can be defined as:

$$a + b := \bigcurlywedge_{c \in A} (a \to \partial c) \to (b \to \partial c) \to \partial c$$

□

THE ARROW TRIPOS Let $\mathcal{A} = (A, \preccurlyeq, \to, S)$ be an arrow algebra.

Definition 4.21. For any set I, the set A^I of functions $I \to A$ can be given an arrow structure by choosing pointwise order and implication, and the *uniform power separator*:[3]

$$S^I := \left\{ \phi : I \to A \;\middle|\; \bigcurlywedge_{i \in I} \phi(i) \in S \right\}$$

makes it into an arrow algebra which we denote as $\mathcal{A}^I$.

The logical order in $\mathcal{A}^I$, which we denote as $\vdash_I$, is therefore given explicitly by:

$$\phi \vdash_I \psi \iff \bigcurlywedge_{i \in I} \phi(i) \to \psi(i) \in S$$

and by Proposition 4.20, $(A^I, \vdash_I)$ is a Heyting prealgebra.

Remark 4.22. As above, given some nucleus j on $\mathcal{A}$, we denote with $\vdash_I^j$ the logical order in $\mathcal{A}_j^I$. Again by the properties of nuclei and separators, $\phi \vdash_I^j \psi$ explicitly means $\phi \vdash_I j\psi$.

3 The name comes from the analog for implicative algebras.

Remark 4.23. In general, $\vdash_I$ is stronger than the pointwise version of $\vdash$: the two coincide only if the separator is closed under arbitrary meets.

However, since the Heyting implication in $(A^I, \vdash_I)$ is the pointwise implication $\to$, it follows that the logical meet $\phi \times \psi$ of $\phi, \psi \in A^I$ can be described as the pointwise logical meet $\theta(i) \coloneqq \phi(i) \times \psi(i)$ of ϕ and ψ. In fact:

$$\begin{aligned} \phi \vdash_I \psi \to \chi &\iff \curlywedge_{i \in I} \phi(i) \to \psi(i) \to \chi(i) \in S \\ &\iff \curlywedge_{i \in I} (\phi(i) \times \psi(i)) \to \chi(i) \in S \\ &\iff \theta \vdash_I \chi \end{aligned}$$

By direct inspection, instead, we also have that the logical join $\phi + \psi$ of $\phi, \psi \in A^I$ can be described as the pointwise logical join $\sigma(i) \coloneqq \phi(i) + \psi(i)$ of ϕ and ψ. Indeed:

- on one hand, $(\lambda xzw.zx)^{\mathcal{A}} \in S$ witnesses $\phi \vdash_I \sigma$, and similarly $(\lambda xzw.wx)^{\mathcal{A}}$ witnesses $\psi \vdash_I \sigma$;
- on the other, if $\phi \vdash_I \chi$ and $\psi \vdash_I \chi$, then $(\lambda z.zs_1 s_2)^{\mathcal{A}} \in S$ witnesses $\sigma \vdash_I \chi$, where $s_1 \coloneqq \curlywedge_{i \in I} \phi(i) \to \partial\chi(i) \in S$ and $s_2 \coloneqq \curlywedge_{i \in I} \psi(i) \to \partial\chi(i) \in S$.

In essence, even though the order in $(A^I, \vdash_I)$ is not determined pointwise by that of $(A, \vdash)$, all the Heyting structure is.

We can finally define the *arrow tripos* induced by $\mathcal{A}$, of which the Heyting prealgebras $(A^I, \vdash_I)$ are the components at each level I.

Theorem 4.24. *Let* $\mathcal{A} = (A, \preccurlyeq, \to, S)$ *be an arrow algebra. The functor:*

$$P_{\mathcal{A}} : \mathsf{Set}^{op} \to \mathsf{HeytPre} \qquad \begin{array}{ccc} I & \longmapsto & (A^I, \vdash_I) \\ {\scriptstyle f}\uparrow & & \downarrow{\scriptstyle -\circ f} \\ J & \longmapsto & (A^J, \vdash_J) \end{array}$$

is a canonically presented tripos.

In particular, left and right adjoints to $f^* : P_{\mathcal{A}}(I) \to P_{\mathcal{A}}(J)$ *are given by:*

$$\begin{aligned} \exists_f(\alpha)(i) &\coloneqq \curlywedge_{a \in A} \left(\curlywedge_{j \in f^{-1}(i)} \alpha(j) \to \partial a \right) \to \partial\partial a \\ \forall_f(\alpha)(i) &\coloneqq \curlywedge_{j \in f^{-1}(i)} \partial\alpha(j) \end{aligned}$$

while a generic element is trivially given by $\mathrm{id}_A : A \to A \in P_{\mathcal{A}}(A)$.

Remark 4.25. If $\mathcal{A}$ is compatible with joins, a left adjoint to $f^* : P_{\mathcal{A}}(I) \to P_{\mathcal{A}}(J)$ can also be defined as:

$$\exists_f(\alpha)(i) := \bigcurlyvee_{j \in f^{-1}(i)} \alpha(j)$$

Remark 4.26. Using the general results on triposes of Chapter 2, (2) in Proposition 4.2 can now be extended to arbitrary propositional formulas as follows.

If ϕ is an intuitionistic tautology built from propositional variables $p_1, \ldots, p_n$, then:

$$P_{\mathcal{A}} \vDash \forall x_1, \ldots, x_n\ \phi(x_1, \ldots, x_n)$$

and hence, by soundness:

$$[\forall x_1, \ldots, x_n\ \phi(x_1, \ldots, x_n)] \in S$$

which gives:

$$\bigcurlywedge_{a_1, \ldots, a_n \in A} \partial\phi(a_i/p_i) \in S$$

In particular, if the main connective of ϕ is $\to$, we can apply (1) of Proposition 4.2, obtaining:

$$\bigcurlywedge_{a_1, \ldots, a_n \in A} \phi(a_i/p_i) \in S$$

As an example, consider De Morgan's laws for disjunction, $\neg(p \vee q) \to \neg p \wedge \neg q$ and $\neg p \wedge \neg q \to \neg(p \vee q)$, which are intuitionistic tautologies. Then:

$$\bigcurlywedge_{a,b \in A} \neg(a + b) \to \neg a \times \neg b \in S$$

$$\bigcurlywedge_{a,b \in A} \neg a \times \neg b \to \neg(a + b) \in S$$

where obviously $\neg x := x \to \bot$.

Notation. We denote with $\mathsf{AT}(\mathcal{A})$ the *arrow topos* induced by $\mathcal{A}$, that is:

$$\mathsf{AT}(\mathcal{A}) := \mathsf{Set}[P_{\mathcal{A}}]$$

4.2 EXAMPLES

IMPLICATIVE ALGEBRAS Implicative algebras can be characterized as arrow algebras where the equality:

$$a \to \bigcurlywedge_{b \in B} b = \bigcurlywedge_{b \in B} a \to b$$

holds for all elements a and subsets B. More precisely, note that in every arrow algebra $\mathcal{A}$ the following hold:

1. $a \to \bigcurlywedge_{b \in B} b \preccurlyeq \bigcurlywedge_{b \in B} a \to b$ for all $a \in A$ and all subsets $B \subseteq A$;
2. $\bigcurlywedge_{b \in B} a \to b \vdash a \to \bigcurlywedge_{b \in B} b$ for all $a \in A$ and all subsets $B \subseteq A$ containing implications only.

Therefore, implicative algebras coincide with arrow algebras where (2) is strengthened to the evidential order and to arbitrary subsets.

FRAMES Every frame $\mathcal{O}(X)$ can be canonically seen as an arrow algebra[4] by using its order and its Heyting implication as the arrow structure, and $\{\top\}$ as the separator. Note then that the logical order coincides with the evidential order, since $x \to y \in S$ if and only if $\top \leqslant x \to y$, which is equivalent to $x \leqslant y$.

Similarly, the logical order $\vdash_I$ on $\mathcal{O}(X)^I$ reduces to the pointwise order, which makes so that the arrow tripos $P_{\mathcal{O}(X)}$ coincides with the localic tripos of Example 2.11, and hence $\mathsf{AT}(\mathcal{O}(X)) \simeq \mathsf{Sh}(\mathcal{O}(X))$.

Remark 4.27. More generally, note that a separator on the canonical arrow structure on $\mathcal{O}(X)$ is precisely a filter on the frame $\mathcal{O}(X)$, but we will not consider any separator other than $\{\top\}$ in this thesis.

PARTIAL COMBINATORY ALGEBRAS A main example of arrow algebras arises from partial combinatory algebras. Let $\mathbb{A} = (A, \leqslant, \cdot, A^\#)$ be a PCA and let DA the set of downward-closed subsets of A. Defining, for $\alpha, \beta \in DA$:

$$\alpha \to \beta := \{\, a \in A \mid a \cdot \alpha{\downarrow} \text{ and } a \cdot \alpha \subseteq \beta \,\}$$

and letting S_{DA} be the family of downward-closed subsets containing an element from the filter $A^\#$, [2, Thm. 3.9] shows how $(DA, \preccurlyeq, \to, S_{DA})$ is an

4 In particular, compatible with joins.

arrow algebra which is compatible with joins; we denote it with $\mathrm{D}\mathbb{A}$ as for the corresponding PCA.[5]

Note then that:

$$\begin{aligned} S_{DA} &= \{\, \alpha \in DA \mid \exists a \in \alpha \cap A^{\#} \,\} \\ &= \{\, \alpha \in DA \mid \exists a \in A^{\#} : \downarrow\{a\} \subseteq \alpha \,\} \\ &= \{\, \alpha \in DA \mid \exists \beta \in T(A^{\#}) : \beta \subseteq \alpha \,\} \end{aligned}$$

meaning that the separator S_{DA} coincides with the filter $(DA)^{\#}$ of Definition 3.5. This makes so that the arrow tripos $P_{D\mathbb{A}}$ coincides with the realizability tripos of Section 3.3, and hence $\mathsf{AT}(\mathrm{D}\,\mathbb{A}) = \mathsf{RT}(\mathbb{A})$.

Remark 4.28. In [2, Thm. 3.10], another construction is identified which yields an arrow algebra starting from a PCA. Although it will not play any role in this thesis, we here introduce it to use it as an example in Chapter 7.

Let $\mathbb{A} = (A, \leqslant, \cdot, A^{\#})$ be a PCA. Then, $\mathbb{A} \times \mathbb{A}$ is a PCA with pointwise order and application, and with the filter $A^{\#} \times A^{\#}$, which means that $\mathrm{D}(\mathbb{A} \times \mathbb{A})$ is an arrow algebra. Explicitly, its elements are downward-closed binary relations on A ordered by inclusion, the implication is defined as:

$$R \to S \coloneqq \{\, (a, a') \in A \times A \mid (a, a') \cdot R{\downarrow} \text{ and } (a, a') \cdot R \subseteq S \,\}$$

while a separator is given by:

$$S_{D(A \times A)} \coloneqq \{\, R \in D(A \times A) \mid \exists (a, a') \in R \cap (A^{\#} \times A^{\#}) \,\}$$

If we restrict to downward-closed binary relations on A which are symmetric and transitive[6], we obtain an arrow algebra which is compatible with joins; we denote it as $\mathrm{PER}(\mathbb{A})$.

5 Similarly, we let $\mathrm{Pow}(\mathbb{A})$ be the arrow algebra $\mathrm{D}\,\mathbb{A}$ for a discrete and absolute PCA $\mathbb{A}$.
6 That is, *partial equivalence relations*.

5 IMPLICATIVE MORPHISMS

After having set up the preliminary material on arrow algebras, the natural step from a categorical point of view is to introduce *morphisms* between them, possibly in such a way as to correspond to transformations between the associated arrow triposes. In this chapter, we introduce the first notion of a morphism between arrow algebras we will see in this thesis, namely that of *implicative morphisms*, and we set up some first results which will be useful in the next chapter.

By definition, an arrow algebra is a poset endowed with an implication and a specified subset: therefore, it would be natural to define morphisms of arrow algebras as monotone functions preserving implications (in a suitable sense) and the specified subset. This intuition, obviously also valid for implicative algebras, is what leads to the definition of *applicative morphisms* in [36]. However, for reasons which will become clear in the next chapter, we will not define our morphisms to be monotone with respect to the evidential order[1], but we will see how this will not actually be an issue. The downside is that, in general, we will have to impose a third condition – automatically satisfied in the case of monotonicity – involving both implications and separators.

5.1 IMPLICATIVE MORPHISMS

Definition 5.1. Let $\mathcal{A} = (A, \preccurlyeq, \to, S_A)$ and $\mathcal{B} = (B, \preccurlyeq, \to, S_B)$ be two arrow algebras. An *implicative morphism* $f : \mathcal{A} \to \mathcal{B}$ is a function $f : A \to B$ satisfying:

i. $f(a) \in S_B$ for all $a \in S_A$;
ii. there exists an element $r \in S_B$ such that $r \preccurlyeq f(a \to a') \to f(a) \to f(a')$ for all $a, a' \in A$;

1 As we have already mentioned, the evidential order is not the most important feature of an arrow algebra anyway.

iii. for any subset $X \subseteq A \times A$,

$$\text{if} \quad \bigcurlywedge_{(a,a')\in X} a \to a' \in S_A \quad \text{then} \quad \bigcurlywedge_{(a,a')\in X} f(a) \to f(a') \in S_B,$$

in which case we say that f is *realized* by $r \in S_B$.

An order 'up to a realizer' can be defined on implicative morphisms as follows. Given two implicative morphisms $f, f' : \mathcal{A} \to \mathcal{B}$, we write $f \vdash f'$ if there exists an element $u \in S_B$ such that $u \preccurlyeq f(a) \to f'(a)$ for all $a \in A$, in which case we say that $f \vdash f'$ is *realized* by u. In other words, this means that:

$$\bigcurlywedge_{a\in A} f(a) \to f'(a) \in S_B$$

i.e. $f \vdash_A f'$ seeing f and f' as elements of the arrow algebra $\mathcal{B}^A$ of Definition 4.21, so in particular it is also equivalent to $f\varphi \vdash_I f'\varphi$ for all sets I and all functions $\varphi : I \to A$.

Remark 5.2. If f is monotone with respect to the evidential order, then (iii) is a consequence of (i) and (ii).

Indeed, given any $X \subseteq A \times A$ such that $\bigcurlywedge_{(a,a')\in X} a \to a' \in S_A$:

– by (ii) we have:

$$\bigcurlywedge_{(a,a')\in X} f(a \to a') \to f(a) \to f(a') \in S_B;$$

– as $f(\bigcurlywedge P) \preccurlyeq \bigcurlywedge f(P)$ for all subsets $P \subseteq A$ by monotonicity, and by (i) and upward-closure of S_B:

$$f\left(\bigcurlywedge_{(a,a')\in X} a \to a'\right) \preccurlyeq \bigcurlywedge_{(a,a')\in X} f(a \to a') \in S_B,$$

from which $\bigcurlywedge_{(a,a')\in X} f(a) \to f(a') \in S_B$ by Proposition 4.2.

Therefore, in proving that a monotone function is an implicative morphism, we will systematically omit to check condition (iii).

Remark 5.3. Applicative morphisms of [36] only satisfy condition (ii) for $a, a' \in A$ such that $a \vdash a'$, while they are monotone by definition. The two notions are hence incomparable in general.

Proposition 5.4. *Arrow algebras, implicative morphisms and their order form a preorder-enriched category* ArrAlg.

Proof. First, let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ be implicative morphisms; let us show that $gf : A \to C$ satisfies the three conditions in Definition 5.1 making it an implicative morphism $\mathcal{A} \to \mathcal{C}$.

i. If $a \in S_A$, then $f(a) \in S_B$ and hence $gf(a) \in S_C$.

ii. By (ii) for f, we know that:

$$\bigcurlywedge_{a,a'} f(a \to a') \to f(a) \to f(a') \in S_B$$

from which, by (iii) for g:

$$\bigcurlywedge_{a,a'} gf(a \to a') \to g(f(a) \to f(a')) \in S_C$$

Moreover, by (ii) for g we know that:

$$\bigcurlywedge_{a,a'} g(f(a) \to f(a')) \to gf(a) \to gf(a') \in S_C$$

from which, by intuitionistic reasoning:

$$\bigcurlywedge_{a,a'} gf(a \to a') \to gf(a) \to gf(a') \in S_C$$

iii. Given $X \subseteq A \times A$ be such that $\bigcurlywedge_{(a,a') \in X} a \to a' \in S_A$, by (iii) for f we have:

$$\bigcurlywedge_{(a,a') \in X} f(a) \to f(a') \in S_B$$

and hence by (iii) for g we have:

$$\bigcurlywedge_{(a,a') \in X} gf(a) \to gf(a') \in S_C$$

Then, for any arrow algebra $\mathcal{A}$, the identity function id_A is an implicative morphism $\mathcal{A} \to \mathcal{A}$, trivially realized by $\mathbf{i} \in S_A$ since we know that:

$$\mathbf{i} \preccurlyeq \bigcurlywedge_{a,a' \in A} (a \to a') \to a \to a'$$

This directly makes ArrAlg into a category.

The fact that $\vdash$ is a preorder on each homset $\mathsf{ArrAlg}(\mathcal{A}, \mathcal{B})$ follows immediately as it is the subpreorder of $(B^A, \vdash_A)$ on implicative morphisms. Therefore, to conclude, we simply need to show that composition of implicative morphisms is order-preserving:

– for $f, f' : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ such that $f \vdash f'$; explicitly, this means that:

$$\bigcurlywedge_{a \in A} f(a) \to f'(a) \in S_B$$

from which, by (iii) in Definition 5.1:

$$\bigcurlywedge_{a \in A} gf(a) \to gf'(a) \in S_C$$

meaning that $gf \vdash gf'$;

– for $f : \mathcal{A} \to \mathcal{B}$ and $g, g' : \mathcal{B} \to \mathcal{C}$, any realizer of $g \vdash g'$ also realizes $gf \vdash g'f$.

□

Example 5.5. Let $j : A \to A$ be a nucleus on an arrow algebra $\mathcal{A}$. Then, (i), (ii) and (vi) in Definition 4.6 immediately imply that j is an implicative morphism $\mathcal{A} \to \mathcal{A}$.

Example 5.6. In a constructive metatheory, truth values are arranged in the frame Ω given by the powerset of the singleton $\{*\}$[2], which we can see as an arrow algebra in the canonical way. For any arrow algebra $\mathcal{A} = (A, \preccurlyeq, \to, S)$, we can then consider the characteristic function of the separator, which is defined constructively as:

$$\chi : A \to \Omega \qquad \chi(a) := \{\, * \mid a \in S \,\}$$

Note that, by upward closure of the separator, χ is monotone. Indeed, if $a \preccurlyeq a'$, to show that $\chi(a) \subseteq \chi(a')$ suppose that $* \in \chi(a)$; then, $a \in S$, hence $a' \in S$ as well, i.e. $* \in \chi(a')$.

We then have that χ is an implicative morphism $\mathcal{A} \to \Omega$.

i. If $a \in S$, then by definition $* \in \chi(a)$, which means that $\chi(a) = \{*\}$.

ii. Let $a, a' \in A$. Then, $\{*\} \subseteq \chi(a \to a') \to \chi(a) \to \chi(a')$ is equivalent to $\chi(a \to a') \subseteq \chi(a) \to \chi(a')$. To show this, suppose $* \in \chi(a \to a')$, meaning that $a \to a' \in S$. So, $\chi(a \to a') = \{*\}$, which means that we can show equivalently that $\chi(a) \subseteq \chi(a')$. Suppose then $* \in \chi(a)$ as well, meaning that $a \in S$; by modus ponens, it follows that $a' \in S$, i.e. $* \in \chi(a')$.

2 Ω is the initial object in the category of frames, and coincides with $2 := \{\bot \leqslant \top\}$ in a classical metatheory.

As promised above, the definition of an implicative morphism can be restated purely in terms of the logical order.

Lemma 5.7. *Let* $\mathcal{A} = (A, \preccurlyeq, \to, S_A)$ *and* $\mathcal{B} = (B, \preccurlyeq, \to, S_B)$ *be arrow algebras. A function* $f : A \to B$ *is an implicative morphism* $\mathcal{A} \to \mathcal{B}$ *if and only if it satisfies:*

1. $\top \vdash f(\top)$;
2. $f(\pi_1 \to \pi_2) \vdash_{A \times A} f\pi_1 \to f\pi_2$, *where* $\pi_1, \pi_2 : A \times A \to A$ *are the two projections;*
3. $f\phi \vdash_I f\psi$ *for any set* I *and all* $\phi, \psi : I \to A$ *such that* $\phi \vdash_I \psi$.

Proof. Condition (2) is a rewriting of condition (ii) recalling that the Heyting implication in $\mathcal{A}^{A \times A}$ is computed pointwise, and condition (3) is a rewriting of condition (iii).

Suppose now f satisfies (i), (ii) and (iii). Then, $f(\top) \in S_B$ since $\top \in S_A$, which means that $\top \vdash f(\top)$.

Conversely, suppose that f satisfies (1), (2) and (3). Note that condition (3) implies that f is monotone with respect to the logical order: therefore, for $a \in S_A$ we have that $\top \vdash a$, and hence $f(\top) \vdash f(a)$. Then, $\top \vdash f(\top)$ implies by modus ponens that $f(\top) \in S_B$, from which $f(a) \in S_B$ as well. □

Corollary 5.8. *If* $f : \mathcal{A} \to \mathcal{B}$ *is an implicative morphism and* $f' : A \to B$ *is such that* $f \dashv\vdash_A f'$ *in* $\mathcal{B}^A$, *then* f' *is an implicative morphism* $\mathcal{A} \to \mathcal{B}$ *as well.*

Many implicative morphisms we will see in the following are monotone with respect to the evidential ordering: as it turns out, this can always be assumed up to isomorphism. Therefore, in principle, we could substitute ArrAlg for an equivalent category where all morphisms are monotone; however, we won't go in this direction, for reasons which will become clear in the next chapter.

Recall that, for any arrow algebra $\mathcal{A}$, we can consider the nucleus $\partial x := \top \to x$, which satisfies $\mathrm{id}_A \dashv\vdash_A \partial$.

Lemma 5.9. *Every implicative morphism is isomorphic to a monotone one.*

Proof. Let $f : \mathcal{A} \to \mathcal{B}$ be an implicative morphism and consider the monotone function:

$$f' : A \to B \qquad f'(a) := \bigcurlywedge_{a \preccurlyeq a'} \partial f(a')$$

Let us show that $f \dashv\vdash_A f'$, which in particular implies that f' is an implicative morphism $\mathcal{A} \to \mathcal{B}$ by the previous corollary.

On one hand, $\partial \vdash_B \mathrm{id}_B$ gives:

$$\curlywedge_a (\partial f(a)) \to f(a) \in S_B$$

from which, since $\curlywedge_{a \preccurlyeq a'} \partial f(a') \preccurlyeq \partial f(a)$ and by upward-closure of S_B:

$$\curlywedge_a (\curlywedge_{a \preccurlyeq a'} \partial f(a')) \to f(a) \in S_B$$

i.e. $f' \vdash_A f$.

On the other hand, $f \vdash_A f'$ explicitly reads as:

$$\curlywedge_a f(a) \to \left(\curlywedge_{a \preccurlyeq a'} \partial f(a') \right) \in S_B$$

Note that, since $\mathbf{a} \in S_B$:

$$\curlywedge_a \left(\curlywedge_{a \preccurlyeq a'} f(a) \to \top \to f(a') \right) \to f(a) \to \left(\curlywedge_{a \preccurlyeq a'} \top \to f(a') \right) \in S_B$$

Therefore, $f \vdash_A f'$ is ensured by Proposition 4.2 if we show:

$$\curlywedge_{(a,a') \in I} f(a) \to \partial f(a') \in S_B$$

where $I := \{ (a, a') \in A \times A \mid a \preccurlyeq a' \}$. By intuitionistic reasoning, this is ensured by:

$$\curlywedge_{(a,a') \in I} f(a) \to f(a') \in S_B \tag{1}$$

$$\curlywedge_{(a,a') \in I} f(a') \to \partial f(a') \in S_B \tag{2}$$

where (1) follows since f is an implicative morphism and $\mathbf{i} \in S_A$ witnesses $\curlywedge_{(a,a') \in I} a \to a' \in S_A$, and (2) since $\mathrm{id}_B \vdash_B \partial$. □

As a matter of fact, the previous construction is pseudofunctorial; this will be useful in Section 7.1.

Proposition 5.10. *For any implicative morphism* $f : \mathcal{A} \to \mathcal{B}$*, let* $Mf : \mathcal{A} \to \mathcal{B}$ *be the monotone implicative morphism* f' *defined in the previous lemma. Then,* M *is a pseudofunctor* $\mathsf{ArrAlg} \to \mathsf{ArrAlg}$.

Proof. Follows immediately since $Mf \dashv\vdash f$ for all implicative morphisms $f : \mathcal{A} \to \mathcal{B}$.

□

5.2 EXAMPLES

Besides the easy examples seen above, let us see the two main classes of implicative morphisms, corresponding to the examples of arrow algebras seen in Section 4.2: those arising from frame homomorphisms, and those arising from morphisms of PCAs.

FRAMES As we know, every frame can be canonically seen as an arrow algebra by choosing $\{\top\}$ as the separator.

Then, any morphism of frames $f : \mathcal{O}(Y) \to \mathcal{O}(X)$, a (necessarily monotone) function preserving finite meets and arbitrary joins, is also an implicative morphism. Indeed:

i. $f(\top) = \top$ as f preserves finite meets;
ii. for $y, y' \in \mathcal{O}(Y)$ we know that $y \wedge (y \to y') \leqslant y'$, so by monotonicity and meet-preservation $f(y) \wedge f(y \to y') \leqslant f(y')$, meaning that $f(y \to y') \leqslant f(y) \to f(y')$ and therefore $\top \leqslant f(y \to y') \to f(y) \to f(y')$.

Remark 5.11. As emerges from the above proof, note more generally that any monotone function $f : \mathcal{O}(Y) \to \mathcal{O}(X)$ which preserves finite meets is an implicative morphism.

Recall moreover that Frm is preorder-enriched by the pointwise order: therefore, given two frame homomorphisms $f, f' : \mathcal{O}(Y) \to \mathcal{O}(X)$, $f \leqslant f'$ in $\mathsf{Frm}(\mathcal{O}(Y), \mathcal{O}(X))$ if and only if $f \vdash f'$ in $\mathsf{ArrAlg}(\mathcal{O}(Y), \mathcal{O}(X))$, since $\vdash_I$ coincides with the pointwise order for any set I. In other words, the inclusion determines a 2-functor $\mathsf{Frm} \to \mathsf{ArrAlg}$ with the additional property that each map $\mathsf{Frm}(\mathcal{O}(Y), \mathcal{O}(X)) \hookrightarrow \mathsf{ArrAlg}(\mathcal{O}(Y), \mathcal{O}(X))$ (preserves and) reflects the order.

While obviously faithful, this inclusion is far from being full. Indeed, consider the initial frame Ω and let $\mathcal{O}(X)$ be a frame such that $\bot \neq \top$: then, the unique frame homomorphism $\Omega \to \mathcal{O}(X)$ is given by $p \mapsto \bigvee\{\top \mid * \in p\}$, whereas the constant function of value $\top$ is an implicative morphism $\Omega \to \mathcal{O}(X)$. As we will see in Section 6.4, frame homomorphisms coincide with *computationally dense* implicative morphisms, while implicative morphisms between frames coincide simply with monotone functions preserving finite meets – hence reversing the previous remark.

PARTIAL COMBINATORY ALGEBRAS Let $\mathbb{A} = (A, \leqslant, \cdot, A^{\#})$ and $\mathbb{B} = (B, \leqslant, \cdot, B^{\#})$ be two PCAs. Let us start by showing how every morphism of PCAs $D\,\mathbb{A} \to D\,\mathbb{B}$ is an implicative morphism with respect to the canonical arrow structures on $D\,\mathbb{A}$ and $D\,\mathbb{B}$.

We warn the reader here that we will see $D\,\mathbb{A}$ and $D\,\mathbb{B}$ simultaneously as PCAs as in Definition 3.5 and as arrow algebras as described in Section 4.2. Both structures possess a notion of application, and in general the two do *not* coincide: for example, given $\alpha, \beta \in DA$, the inequality

$$(\alpha \to \beta) \cdot \alpha \subseteq \beta$$

only holds with respect to the application defining the partial applicative structure on $D\,\mathbb{A}$, while it does not hold for the arrow-algebraic application of Definition 4.11.[3] In this thesis, we will not make any use of the latter form of application when dealing with arrow algebras of the form $D\,\mathbb{A}$; hence, the application considered will always be the one defining the partial applicative structure.

Lemma 5.12. *Let* $f : D\,\mathbb{A} \to D\,\mathbb{B}$ *be a morphism of PCAs. Then,* f *is also an implicative morphism* $D\,\mathbb{A} \to D\,\mathbb{B}$.

Proof. Let us verify the three conditions in Definition 5.1.

i. Condition (i) is ensured by (i) in Definition 3.7 since $S_{DA} = (DA)^{\#}$ and $S_{DB} = (DB)^{\#}$.

ii. To show condition (ii), we need to find some $\rho \in (DB)^{\#}$ such that $\rho \subseteq f(\alpha \to \beta) \to f(\alpha) \to f(\beta)$ for all $\alpha, \beta \in DA$. By definition, recall that:

   – there exists $\tau \in (DB)^{\#}$ such that if $\alpha\alpha'{\downarrow}$, then $\tau f(\alpha) f(\alpha'){\downarrow}$ and $\tau f(\alpha) f(\alpha') \subseteq f(\alpha\alpha')$;

   – there exists $\upsilon \in (DB)^{\#}$ such that if $\alpha \subseteq \alpha'$ then $\upsilon f(\alpha){\downarrow}$ and $\upsilon f(\alpha) \subseteq f(\alpha')$.

   By combinatory completeness, consider then:

$$\rho := (\lambda^* v, w.\upsilon(\tau v w)) \in (DB)^{\#}$$

   Since $(\alpha \to \beta) \cdot \alpha{\downarrow}$, we know that:

$$\tau\, f(\alpha \to \beta)\, f(\alpha){\downarrow} \quad \text{and} \quad \tau\, f(\alpha \to \beta)\, f(\alpha) \subseteq f((\alpha \to \beta) \cdot \alpha)$$

3 For this other application, we only have the inequality $(\alpha \to \partial\beta) \cdot \alpha \subseteq \partial\beta$, where $\partial\beta := A \to \beta$.

Since moreover $(\alpha \to \beta) \cdot \alpha \subseteq \beta$, we also know that:

$$\upsilon\ f((\alpha \to \beta) \cdot \alpha)\downarrow \quad \text{and} \quad \upsilon\ f((\alpha \to \beta) \cdot \alpha) \subseteq f(\beta)$$

So, by downward-closure of the domain of the application:

$$\upsilon(\tau\ f(\alpha \to \beta)\ f(\alpha))\downarrow \quad \text{and} \quad \upsilon(\tau\ f(\alpha \to \beta)\ f(\alpha)) \subseteq f(\beta)$$

Therefore:

$$\rho\ f(\alpha \to \beta)\ f(\alpha)\downarrow \quad \text{and} \quad \rho\ f(\alpha \to \beta)\ f(\alpha) \subseteq f(\beta)$$

or, in other words:

$$\rho \subseteq f(\alpha \to \beta) \to f(\alpha) \to f(\beta)$$

iii. Let $X \subseteq DA \times DA$ be such that there exists some $\sigma \in (DA)^{\#}$ satisfying $\sigma \subseteq \alpha \to \beta$ for all $(\alpha, \beta) \in X$. To show condition (iii), we need to find some $\rho \in (DB)^{\#}$ such that $\rho \subseteq f(\alpha) \to f(\beta)$ for all $(\alpha, \beta) \in X$.

By combinatory completeness, since $f(\sigma) \in (DB)^{\#}$ by condition (i), consider then:

$$\rho := (\lambda^* w.\upsilon(\tau f(\sigma) w)) \in (DB)^{\#}$$

Since $\sigma \cdot \alpha\downarrow$ and $\sigma \cdot \alpha \subseteq \beta$, exactly as above we have that:

$$\upsilon(\tau\ f(\sigma)\ f(\alpha))\downarrow \quad \text{and} \quad \upsilon(\tau\ f(\sigma)\ f(\alpha)) \subseteq f(\beta)$$

Therefore:

$$\rho\ f(\alpha)\downarrow \quad \text{and} \quad \rho\ f(\alpha) \subseteq f(\beta)$$

or, in other words:

$$\rho \subseteq f(\alpha) \to f(\beta)$$

□

Remark 5.13. The two orders also coincide, by definition of the implication in downsets PCAs: given two morphisms of PCAs $f, f' : \mathrm{D}\,\mathbb{A} \to \mathrm{D}\,\mathbb{B}$, then $f \leqslant f'$ in $\mathsf{OPCA}(\mathrm{D}\,\mathbb{A}, \mathrm{D}\,\mathbb{B})$ if and only if $f \vdash f'$ in $\mathsf{ArrAlg}(\mathrm{D}\,\mathbb{A}, \mathrm{D}\,\mathbb{B})$.

Let now $f : \mathbb{A} \to \mathbb{B}$ be a partial applicative morphism, i.e. a morphism of PCAs $\mathbb{A} \to \mathrm{D}\,\mathbb{B}$. Recall that f corresponds essentially uniquely to the D-algebra morphism:

$$\widetilde{f} : \mathrm{D}\,\mathbb{A} \to \mathrm{D}\,\mathbb{B} \qquad \widetilde{f}(\alpha) := \bigcup_{a \in \alpha} f(a)$$

and the assignment $f \mapsto \widetilde{f}$ is 2-functorial on $\mathsf{OPCA_D}$. Together with the previous lemma and remark, this immediately implies the following.

Proposition 5.14. *The assignment* $f \mapsto \widetilde{f}$ *determines a 2-functor:*

$$\widetilde{\mathrm{D}} : \ \mathsf{OPCA_D} \longrightarrow \mathsf{ArrAlg}$$

Moreover, for any PCAs $\mathbb{A}$ *and* $\mathbb{B}$*, the map:*

$$\mathsf{OPCA_D}(\mathbb{A}, \mathbb{B}) \longrightarrow \mathsf{ArrAlg}(\mathrm{D}\,\mathbb{A}, \mathrm{D}\,\mathbb{B})$$

(preserves and) reflects the order.

Remark 5.15. The maps $\mathsf{OPCA_D}(\mathbb{A}, \mathbb{B}) \to \mathsf{ArrAlg}(\mathrm{D}\,\mathbb{A}, \mathrm{D}\,\mathbb{B})$ defined by $\widetilde{\mathrm{D}}$ are obviously not essentially surjective, meaning that $\widetilde{\mathrm{D}}$ is not 2-fully faithful. Indeed, any morphism of PCAs $\mathrm{D}\,\mathbb{A} \to \mathrm{D}\,\mathbb{B}$ is an implicative morphism $\mathrm{D}\,\mathbb{A} \to \mathrm{D}\,\mathbb{B}$, but obviously only those which are union-preserving are D-algebra morphisms and therefore arise as $\widetilde{\mathrm{D}}f$ for some partial applicative morphism $f : \mathbb{A} \to \mathbb{B}$. We will see more about the interplay of these notions in Section 6.4.

Question. How does the PER(−) construction behave with respect to (partial applicative) morphisms of PCAs and implicative morphisms?

6 ARROW TRIPOSES

We have finally arrived at the heart of this thesis. In this chapter, we further the study of implicative morphisms and their relations with transformations of arrow triposes, lifting the association $\mathcal{A} \mapsto P_{\mathcal{A}}$ to a (2-)functor defined on a suitable category of arrow algebras. The main goals, in this perspective, are the following.

i. First, we will characterize implicative morphisms $\mathcal{A} \to \mathcal{B}$ as those functions $A \to B$ which induce by postcomposition a left exact transformation of arrow triposes $P_{\mathcal{A}} \to P_{\mathcal{B}}$.

ii. Then, we will determine a suitable notion of *computational density* which characterizes those implicative morphisms $\mathcal{A} \to \mathcal{B}$ such that the induced transformation $P_{\mathcal{A}} \to P_{\mathcal{B}}$ has a right adjoint, hence corresponding to geometric morphism of triposes $P_{\mathcal{B}} \to P_{\mathcal{A}}$.

iii. Finally, we will specify the previous correspondence to the case of geometric inclusions $P_{\mathcal{B}} \hookrightarrow P_{\mathcal{A}}$ and see how they correspond to nuclei on $\mathcal{A}$.

To motivate this, let us reconsider the examples seen in the previous chapter.

FRAMES The localic tripos $P_{\mathcal{O}(X)}$ over a frame $\mathcal{O}(X)$ coincides with the arrow tripos obtained seeing $\mathcal{O}(X)$ as an arrow algebra in the canonical way. By [35] we know that any frame homomorphism $f^* : \mathcal{O}(X) \to \mathcal{O}(Y)$ induces a geometric morphism of triposes $P_{\mathcal{O}(Y)} \to P_{\mathcal{O}(X)}$ whose inverse image is given by postcomposition with f^* at each component; as we have shown in the previous chapter, f^* is an implicative morphism $\mathcal{O}(X) \to \mathcal{O}(Y)$.

Moreover, the direct image is given by postcomposition with $f_* : \mathcal{O}(Y) \to \mathcal{O}(X)$ at each component, where f_* is the right adjoint of f^* as maps of posets – which always exists for frame homomorphisms.

PARTIAL COMBINATORY ALGEBRAS The realizability tripos $P_{\mathbb{A}}$ over a PCA $\mathbb{A}$ coincides with the arrow tripos $P_{D\,\mathbb{A}}$ obtained seeing $D\,\mathbb{A}$ as an arrow algebra in the canonical way. As we have seen in Chapter 3, any partial applicative morphism $f : \mathbb{A} \to \mathbb{B}$ induces a left exact transformation of triposes $P_{\mathbb{A}} \to P_{\mathbb{B}}$ given by postcomposition with $\widetilde{f} : DA \to DB$ at each component; as we've shown in the previous chapter, $\widetilde{f}$ is an implicative morphism $D\,\mathbb{A} \to D\,\mathbb{B}$.

Moreover, the induced left exact transformation admits a right adjoint making it the inverse image of a geometric morphism $P_{\mathbb{B}} \to P_{\mathbb{A}}$ if and only if f is computationally dense, in which case the direct image is given by postcomposition with $h : D\,\mathbb{B} \to D\,\mathbb{A}$, where h is the right adjoint of $\widetilde{f}$ in OPCA – which is equivalent to the computational density of f.

6.1 LEFT EXACT TRANSFORMATIONS OF ARROW TRIPOSES

Let us start with a lemma we will make use of in the following. Recall that, given two arrow algebras $\mathcal{A} = (A, \preccurlyeq, \to, S_A)$ and $\mathcal{B} = (B, \preccurlyeq, \to, S_B)$, for every set I we can consider the arrow algebras $\mathcal{A}^I = (A^I, \preccurlyeq, \to, S_A^I)$ and $\mathcal{B}^I = (B^I, \preccurlyeq, \to, S_B^I)$ as in Definition 4.21.

Lemma 6.1. *Let $f : \mathcal{A} \to \mathcal{B}$ be an implicative morphism.*
For any set I, $f^I := f \circ -$ is an implicative morphism $\mathcal{A}^I \to \mathcal{B}^I$.

Proof. Let us verify the three conditions in Definition 5.1.

i. To show condition (i), recall that we can equivalently prove that $f^I(\top_I) \in S_B^I$, where $\top_I : I \to A$ is the constant function of value $\top \in A$. Note then that by condition (i) for f we have:

$$\bigcurlywedge_i f(\top_I(i)) = \bigcurlywedge_i f(\top) = f(\top) \in S_B$$

meaning that $f^I(\top) \in S_B^I$.

ii. Let $r \in S_B$ be a realizer for f and let $\rho : I \to B$ be the constant function at r; as $\bigcurlywedge_i \rho(i) = r \in S_B$, we know that $\rho \in S_B^I$.

Then, for any $\phi, \phi' \in A^I$:

$$r \preccurlyeq f(\phi(i) \to \phi'(i)) \to f\phi(i) \to f\phi'(i) \quad \forall i \in I$$

i.e., since the order and implications are defined pointwise in $\mathcal{B}^I$:

$$\rho \preccurlyeq f^I(\phi \to \phi') \to f^I\phi \to f^I\phi'$$

meaning that ρ realizes f^I.

iii. Let $X \subseteq A^I \times A^I$ be such that $\curlywedge_{(\phi,\psi)\in X} \phi \to \psi \in S^I_A$. For the sake of notation, assume $X = \{ (\phi_j, \psi_j) \mid j \in J \}$; then, since the order (hence meets) and implications are defined pointwise in $\mathcal{A}^I$, we have:

$$\curlywedge_i \curlywedge_j \phi_j(i) \to \psi_j(i) \in S_A$$

from which, by (iii) for f:

$$\curlywedge_i \curlywedge_j f\phi_j(i) \to f\psi_j(i) \in S_A$$

meaning that $\curlywedge_j f^I\phi_j \to f^I\psi_j \in S^I_A$.

□

Fix now an implicative morphism $f : \mathcal{A} \to \mathcal{B}$ and define the transformation:

$$\Phi^+_f : P_\mathcal{A} \to P_\mathcal{B} \qquad (\Phi^+_f)_I(\phi) := f^I\phi = f \circ \phi$$

Indeed, monotonicity of each component $(\Phi^+_f)_I : P_\mathcal{A}(I) \to P_\mathcal{B}(I)$ precisely corresponds to condition (iii) in Definition 5.1, while naturality is obvious.

Let us now show that, for every set I, $(\Phi^+_f)_I : P_\mathcal{A}(I) \to P_\mathcal{B}(I)$ preserves finite meets up to isomorphism. As $f^I(\top_I) \in S^I_B$ we know that $f^I(\top_I) \dashv\vdash_I \top_I$, so we only have to show that for any $\phi, \psi \in A^I$:

$$f^I(\phi \times \psi) \dashv\vdash_I f^I\phi \times f^I\psi$$

where $f^I\phi \times f^I\psi$ is the meet of $f^I\phi$ and $f^I\psi$ in $P_\mathcal{B}(I)$ which, we recall from Remark 4.23, can be assumed to be defined pointwise.

Of course $f^I(\phi \times \psi) \vdash_I f^I\phi \times f^I\psi$ follows simply by monotonicity of f^I with respect to the logical order; on the other hand, $f^I\phi \times f^I\psi \vdash_I f^I(\phi \times \psi)$ is ensured by the following lemma applied to $f^I : \mathcal{A}^I \to \mathcal{B}^I$.

Lemma 6.2. *Let $f : \mathcal{A} \to \mathcal{B}$ be an implicative morphism. Then:*

$$\curlywedge_{a,b} (f(a) \times f(b)) \to f(a \times b) \in S_B$$

Proof. First, recall that:

$$\curlywedge_{a,b} a \to b \to (a \times b) \in S_A$$

from which, by (iii) in Definition 5.1:

$$\bigcurlywedge_{a,b} f(a) \to f(b \to (a \times b)) \in S_B$$

whereas, by (ii):

$$\bigcurlywedge_{a,b} f(b \to (a \times b)) \to f(b) \to f(a \times b) \in S_B$$

so by intuitionistic reasoning we conclude:

$$\bigcurlywedge_{a,b} f(a) \to f(b) \to f(a \times b) \in S_B$$

which means:

$$\bigcurlywedge_{a,b} (f(a) \times f(b)) \to f(a \times b) \in S_B$$

□

Summing up, we have shown the following.

Proposition 6.3. *Let* $f : \mathcal{A} \to \mathcal{B}$ *be an implicative morphism. Then:*

$$\Phi_f^+ : P_{\mathcal{A}} \to P_{\mathcal{B}} \qquad (\Phi_f^+)_I(\phi) \coloneqq f \circ \phi$$

is a left exact transformation of triposes.

As promised above, we can also prove the converse: up to isomorphism, every left exact transformation of arrow triposes is induced in the sense of the previous proposition by an implicative morphism which is unique up to isomorphism.

Proposition 6.4. *The association* $f \mapsto \Phi_f^+$ *determines a 2-fully faithful 2-functor:*

$$\mathsf{ArrAlg} \longrightarrow \mathsf{Trip}_{\mathsf{lex}}(\mathsf{Set})$$

Explicitly, this means that for any arrow algebras $\mathcal{A}$ *and* $\mathcal{B}$ *there is an equivalence of preorder categories:*

$$\mathsf{ArrAlg}(\mathcal{A}, \mathcal{B}) \simeq \mathsf{Trip}_{\mathsf{lex}}(\mathsf{Set})(P_{\mathcal{A}}, P_{\mathcal{B}})$$

Proof. By the previous discussion, we have a functor $\mathsf{ArrAlg} \to \mathsf{Trip}_{\mathsf{lex}}(\mathsf{Set})$; 2-functoriality amounts to showing that given any two implicative morphisms $f, f' : \mathcal{A} \to \mathcal{B}$ such that $f \vdash f'$, then $\Phi_f^+ \leqslant \Phi_{f'}^+$. By definition, $f \vdash f'$ means that $f\phi \vdash_I f'\phi$ holds for any set I and any $\phi : I \to A$, i.e. $(\Phi_f^+)_I(\phi) \vdash_I (\Phi_{f'}^+)_I(\phi)$ holds in $P_{\mathcal{B}}(I)$, which means means that $\Phi_f^+ \leqslant \Phi_{f'}^+$.

Note then that the converse also holds: if $\Phi^+_f \leqslant \Phi^+_{f'}$, then in particular $(\Phi^+_f)_A(\mathrm{id}_A) \vdash_A (\Phi^+_{f'})_A(\mathrm{id}_A)$, which means that $f \vdash f'$.

Let now $\Phi^+ : P_{\mathcal{A}} \to P_{\mathcal{B}}$ be any left exact transformation of arrow triposes. Recall that, up to isomorphism, Φ^+ is given by postcomposition with the function:

$$f \coloneqq (\Phi^+)_A(\mathrm{id}_A) : A \to B$$

Let us now verify that f satisfies the three conditions[1] in Definition 5.1.

i. To show condition (i), recall that we can equivalently prove that $f(\top) \in S_B$. By left exactness we know that $(\Phi^+)_I(\top_I) \dashv\vdash_I \top_I$, which for $I \coloneqq 1$ means that $f(\top) \dashv\vdash \top$, i.e. $f(\top) \in S_B$.

ii. Let $I \coloneqq A \times A$; recall that condition (ii) can be rewritten as:

$$f(\pi_1 \to \pi_2) \vdash_I f\pi_1 \to f\pi_2$$

where $\pi_1, \pi_2 : I \to A$ are the two projections. In terms of Φ^+, this means that we have to show:

$$(\Phi^+)_I(\pi_1 \to \pi_2) \vdash_I (\Phi^+)_I(\pi_1) \to (\Phi^+)_I(\pi_2)$$

Through the Heyting adjunction in $P_{\mathcal{B}}(I)$, the previous is equivalent to:

$$(\Phi^+)_I(\pi_1 \to \pi_2) \times (\Phi^+)_I(\pi_1) \vdash_I (\Phi^+)_I(\pi_2)$$

i.e., by left exactness:

$$(\Phi^+)_I(\pi_1 \to \pi_2 \times \pi_1) \vdash_I (\Phi^+)_I(\pi_2)$$

which is ensured by monotonicity since $\pi_1 \to \pi_2 \times \pi_1 \vdash_I \pi_2$.

iii. Condition (iii) precisely corresponds to the monotonicity of each component $(\Phi^+)_I$.

Therefore, the association $\Phi^+ \mapsto (\Phi^+)_A(\mathrm{id}_A)$ realizes the desired inverse equivalence since obviously $(\Phi^+_f)_A(\mathrm{id}_A) = f$ for all implicative morphisms $f : \mathcal{A} \to \mathcal{B}$. □

Remark 6.5. Recall by [29] that[2] every Set-based tripos is isomorphic to an implicative one, and hence to an arrow one. Therefore, the 2-functor $\mathsf{ArrAlg} \to \mathsf{Trip}_{\mathsf{lex}}(\mathsf{Set})$ is actually a 2-equivalence of 2-categories.

1 If we assumed implicative morphisms to be monotone, we would not be able to prove that f is one.

2 At least, assuming the Axiom of Choice.

6.2 GEOMETRIC MORPHISMS OF ARROW TRIPOSES

RIGHT ADJOINTS AND GEOMETRIC MORPHISMS Let us now move to geometric morphisms: as we will see in a moment, the existence of a right adjoint at the level of transformations of triposes exactly corresponds to the existence of a right adjoint in ArrAlg.

Definition 6.6. An implicative morphism $f : \mathcal{A} \to \mathcal{B}$ is *computationally dense*[3] if it admits a right adjoint in ArrAlg, that is, if there exists an implicative morphism $h : \mathcal{B} \to \mathcal{A}$ such that $fh \vdash \mathrm{id}_B$ and $\mathrm{id}_A \vdash hf$.

For any arrow algebra $\mathcal{A}$, the identity $\mathrm{id}_A : \mathcal{A} \to \mathcal{A}$ is computationally dense, as it is trivially right adjoint to itself. The following lemma, therefore, allows us to define the wide subcategory $\mathsf{ArrAlg}_{\mathsf{cd}}$ of ArrAlg on computationally dense morphisms.

Lemma 6.7. *Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ be implicative morphisms. If f and g are computationally dense, then so is gf.*

Proof. Let $h : \mathcal{B} \to \mathcal{A}$ be right adjoint to f and let $h' : \mathcal{C} \to \mathcal{B}$ be right adjoint to g. Let us show that $hh' : \mathcal{C} \to \mathcal{A}$ is right adjoint to gf.

- On one hand, we know that $fh \vdash_B \mathrm{id}_B$; in particular, $fhh' \vdash_C h'$. Then, $gfhh' \vdash_C gh'$ as g is an implicative morphism, and hence $gfhh' \vdash_C \mathrm{id}_C$ since $gh' \vdash_C \mathrm{id}_C$.
- On the other, we know that $\mathrm{id}_B \vdash_B h'g$; in particular, $f \vdash_A h'gf$. Then, $hf \vdash_A hh'gf$ as h is an implicative morphism, and hence $\mathrm{id}_A \vdash_A hh'gf$ since $\mathrm{id}_A \vdash hf$.

□

Fix now a computationally dense implicative morphism $f : \mathcal{A} \to \mathcal{B}$ with right adjoint $h : \mathcal{B} \to \mathcal{A}$ and consider the left exact transformation induced by h as in Proposition 6.3:

$$\Phi_+ : P_{\mathcal{B}} \to P_{\mathcal{A}} \qquad (\Phi_+)_I(\phi) := h \circ \phi$$

Lemma 6.8. *For every set I, $(\Phi_+)_I : P_{\mathcal{B}}(I) \to P_{\mathcal{A}}(I)$ is right adjoint to the map $(\Phi_f^+)_I : \psi \mapsto f \circ \psi$.*

3 The name is obviously taken from the theory of PCAs, and it is also used in [36] in the context of applicative morphisms.

Proof. By the universal property of the counit of the adjunction, it suffices to show that for all $\phi : I \to B$:

1. $fh\phi \vdash_I \phi$ in $P_{\mathcal{B}}(I)$;

2. for any $\psi : I \to A$ such that $f\psi \vdash_I \phi$ in $P_{\mathcal{B}}(I)$, then $\psi \vdash_I h\phi$ in $P_{\mathcal{A}}(I)$.

(1) clearly follows as h is right adjoint to f. To show (2), instead, suppose $f\psi \vdash_I \phi$; then, $hf\psi \vdash_I h\phi$ as h is an implicative morphism, and hence $\psi \vdash_I h\psi$ since $\mathrm{id}_A \vdash_A hf$. □

The results of the previous section then immediately yield the following.

Theorem 6.9. *Let* $f : \mathcal{A} \to \mathcal{B}$ *be a computationally dense implicative morphism with right adjoint* $h : \mathcal{B} \to \mathcal{A}$*. Then:*

$$P_{\mathcal{B}} \underset{\Phi_+}{\overset{\Phi^+}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} P_{\mathcal{A}} \qquad \begin{cases} (\Phi^+)_I(\psi) \coloneqq f \circ \psi \\ (\Phi_+)_I(\phi) \coloneqq h \circ \psi \end{cases}$$

is a geometric morphism of triposes.

As we did for implicative morphisms and left exact transformations, in this case too we can prove the converse: up to isomorphism, every geometric morphism of arrow triposes is induced by an essentially unique computationally dense implicative morphism.

Proposition 6.10. *The 2-functor of Proposition 6.4 restricts to a 2-fully faithful 2-functor:*

$$\mathsf{ArrAlg}_{\mathsf{cd}} \longrightarrow \mathsf{Trip}_{\mathsf{geo}}(\mathsf{Set})$$

Explicitly, this means that for any arrow algebras $\mathcal{A}$ *and* $\mathcal{B}$ *there is an equivalence of preorder categories:*

$$\mathsf{ArrAlg}_{\mathsf{cd}}(\mathcal{A}, \mathcal{B}) \simeq \mathsf{Trip}_{\mathsf{geo}}(\mathsf{Set})(P_{\mathcal{A}}, P_{\mathcal{B}})$$

Proof. By the previous section and the previous discussion, we have a 2-functor $\mathsf{ArrAlg}_{\mathsf{cd}} \to \mathsf{Trip}_{\mathsf{geo}}(\mathsf{Set})$ such that, given any two computationally dense implicative morphisms $f, f' : \mathcal{A} \to \mathcal{B}$, $f \vdash f'$ if and only if $\Phi^+_f \leqslant \Phi^+_{f'}$.

Let now $\Phi^+ : P_{\mathcal{A}} \to P_{\mathcal{B}}$ be a left exact transformation of triposes having a right adjoint $\Phi_+ : P_{\mathcal{B}} \to P_{\mathcal{A}}$. Recall that, up to isomorphism, Φ^+ is given by postcomposition with $f \coloneqq (\Phi^+)_A(\mathrm{id}_A) : A \to B$, which is an implicative morphism $\mathcal{A} \to \mathcal{B}$ by Proposition 6.4. In the same way, as it is also left exact, Φ_+ is given up to isomorphism by postcomposition with the implicative morphism $h \coloneqq (\Phi_+)_B(\mathrm{id}_B) : \mathcal{B} \to \mathcal{A}$. Moreover, the adjunction between Φ^+ and Φ_+ directly yields $fh \vdash \mathrm{id}_B$ and $\mathrm{id}_A \vdash hf$, meaning that h is right adjoint to f making it computationally dense.

□

Remark 6.11. As in Remark 6.5, the 2-functor $\mathsf{ArrAlg}_{\mathsf{cd}} \to \mathsf{Trip}_{\mathsf{geo}}(\mathsf{Set})$ is a 2-equivalence of 2-categories.[4]

EQUIVALENCES With usual 2-categorical notation, we say that an implicative morphism $f : \mathcal{A} \to \mathcal{B}$ is an *equivalence* if there exists another implicative morphism $g : \mathcal{B} \to \mathcal{A}$ such that $fg \dashv\vdash \mathrm{id}_B$ in $\mathsf{ArrAlg}(\mathcal{B}, \mathcal{B})$ and $gf \dashv\vdash \mathrm{id}_A$ in $\mathsf{ArrAlg}(\mathcal{A}, \mathcal{A})$, in which case g is a *quasi-inverse* of f. Two arrow algebras are then *equivalent* if there exists an equivalence between them; clearly, equivalent arrow algebras induce equivalent triposes.

Lemma 6.12. *Let $f : \mathcal{A} \to \mathcal{B}$ be an equivalence of arrow algebras.*

Then, f is computationally dense, and the induced geometric morphism of triposes $\Phi : P_{\mathcal{B}} \to P_{\mathcal{A}}$ is an equivalence.

Proof. Let $g : \mathcal{B} \to \mathcal{A}$ be a quasi-inverse of f. As g is in particular right adjoint to f in ArrAlg, f is computationally dense, and the induced geometric morphism $\Phi : P_{\mathcal{B}} \to P_{\mathcal{A}}$ is given by:

$$(\Phi^+)_I(\psi) = f \circ \psi \qquad (\Phi_+)_I(\phi) = g \circ \phi$$

In particular, $\Phi^+\Phi_+$ and $\Phi_+\Phi^+$ are isomorphic to identities as $fg \dashv\vdash \mathrm{id}_B$ and $gf \dashv\vdash \mathrm{id}_A$, meaning that Φ is an equivalence. □

By the previous results, we can also easily address the converse.

Proposition 6.13. *Let $\Phi : P_{\mathcal{A}} \to P_{\mathcal{B}}$ be an equivalence of arrow triposes. Then, Φ is induced up to isomorphism by an (essentially unique) equivalence of arrow algebras $f : \mathcal{A} \to \mathcal{B}$.*

Proof. Let $\Psi : P_{\mathcal{B}} \to P_{\mathcal{A}}$ be a quasi-inverse of Φ. Then, Φ is both left and right adjoint to Ψ, which means in particular that the pair (Φ, Ψ) defines a geometric morphism $P_{\mathcal{B}} \to P_{\mathcal{A}}$. Therefore, by Proposition 6.10, Φ is induced up to isomorphism by an (essentially unique) computationally dense implicative morphism $f : \mathcal{A} \to \mathcal{B}$; a right adjoint $g : \mathcal{B} \to \mathcal{A}$ inducing

4 Again, assuming the Axiom of Choice.

Ψ up to isomorphism then satisfies $fg \dashv\vdash \mathrm{id}_B$ and $gf \dashv\vdash \mathrm{id}_A$, making f an equivalence. $\square$

Remark 6.14. We say that an arrow algebra $\mathcal{A}$ is *trivial* if $S = A$, or equivalently if $\bot \in S$.

It is then immediate to show that $\mathcal{A}$ is trivial if and only if the unique map $\mathcal{A} \to \{*\}$ – which is obviously an implicative morphism – is an equivalence. Hence, $\mathcal{A}$ is trivial if and only if $\mathsf{AT}(\mathcal{A})$ is (equivalent to) the trivial topos.

6.3 Inclusions and surjections

Let us start this section by characterizing inclusions and surjections of arrow triposes.

Recall that a geometric morphism of arrow triposes $\Phi : P_{\mathcal{B}} \to P_{\mathcal{A}}$ is an *inclusion* if $(\Phi_+)_I$ reflects the order for any set I, or equivalently if $(\Phi^+)_I(\Phi_+)_I(\phi) \dashv\vdash_I \phi$ for any set I and any $\phi : I \to B$. Dually, Φ is a *surjection* if $(\Phi^+)_I$ reflects the order for any set I, or equivalently if $(\Phi_+)_I(\Phi^+)_I(\phi) \dashv\vdash_I \phi$ for any set I and any $\phi : I \to A$.

Recall moreover the following general definition.

Definition 6.15. Let C be a preorder-enriched category. An arrow $f : A \to B$ in C is a *lax epimorphism* if, for all $C \in \mathsf{C}$, the map

$$- \circ f : \mathsf{C}(B, C) \to \mathsf{C}(A, C)$$

is fully-faithful as a functor between preorder categories, which explicitly means that $p \leqslant q$ for all $p, q : B \to C$ such that $pf \leqslant qf$.

Dually, f is a *lax monomorphism* if, for all $C \in \mathsf{C}$, the map

$$f \circ - : \mathsf{C}(C, A) \to \mathsf{C}(C, B)$$

is fully-faithful as a functor between preorder categories, which explicitly means that $p \leqslant q$ for all $p, q : C \to A$ such that $fp \leqslant fq$.

Definition 6.16. A computationally dense implicative morphism $f : \mathcal{A} \to \mathcal{B}$ is an *implicative surjection* (resp. *implicative injection*) if it is a lax epimorphism (resp. lax monomorphism) in ArrAlg.

Proposition 6.17. *Let $f : \mathcal{A} \to \mathcal{B}$ be a computationally dense implicative morphism with right adjoint $h : \mathcal{B} \to \mathcal{A}$ and let $\Phi : P_{\mathcal{B}} \to P_{\mathcal{A}}$ be the induced geometric morphism of arrow triposes. The following are equivalent:*

1. *Φ is an inclusion;*

2. $fh \dashv\vdash_B \mathrm{id}_B$*;*

3. f *is an implicative surjection.*

Dually, the following are equivalent:

1. *Φ is a surjection;*

2. $hf \dashv\vdash_A \mathrm{id}_A$*;*

3. f *is an implicative injection.*

Proof. For (1) $\Leftrightarrow$ (2), recall that the inverse image Φ^+ is given by postcomposition with f, and the direct image Φ_+ is given by postcomposition with h: therefore, Φ is an inclusion if and only if $fh\phi \dashv\vdash_I \phi$ for any set I and any $\phi \in P_{\mathcal{B}}(I)$, which is equivalent to $fh \dashv\vdash_B \mathrm{id}_B$.

For (2) $\Rightarrow$ (3), suppose $p, q : \mathcal{B} \to \mathcal{C}$ are such that $pf \vdash qf$. Then, $pfh \vdash qfh$, and hence $p \vdash q$.

For (3) $\Rightarrow$ (2), of course $fh \vdash \mathrm{id}_B$; conversely, to show that $\mathrm{id}_B \vdash fh$ it then suffices to show that $f \vdash fhf$, which is ensured by $\mathrm{id}_A \vdash hf$. □

Corollary 6.18. *For any arrow algebras $\mathcal{A}$ and $\mathcal{B}$, there are equivalences of preorder categories between:*

- *implicative surjections $\mathcal{A} \to \mathcal{B}$ and geometric inclusions $P_{\mathcal{B}} \hookrightarrow P_{\mathcal{A}}$;*
- *implicative injections $\mathcal{A} \to \mathcal{B}$ and geometric surjections $P_{\mathcal{B}} \twoheadrightarrow P_{\mathcal{A}}$.*

Proof. Combining Proposition 6.10 with the previous proposition. □

Remark 6.19. Of course, an implicative morphism is an equivalence if and only if it is both an implicative surjection and an implicative inclusion.

NUCLEI AND SUBTRIPOSES Let $\mathcal{A} = (A, \preccurlyeq, \to, S)$ be an arrow algebra.

As we have seen in Proposition 4.7, every nucleus j on $\mathcal{A}$ determines a new arrow algebra $\mathcal{A}_j = (A, \preccurlyeq, \to_j, S_j)$ where:

$$a \to_j b := a \to jb \qquad S_j := \{\, a \in A \mid ja \in S \,\}$$

With the previous machinery, [2, Prop. 6.3] can then be reduced to the following observation.

Lemma 6.20. id_A *is an implicative surjection $\mathcal{A} \to \mathcal{A}_j$, with j as a right adjoint.*

Proof. Let us start by showing that id_A is an implicative morphism $\mathcal{A} \to \mathcal{A}_j$. As the evidential order is the same in $\mathcal{A}$ and $\mathcal{A}_j$, we only have to verify (i) and (ii) in Definition 5.1.

i. If $a \in S$, then $ja \in S$, meaning that $a \in S_j$.

ii. Condition (ii) explicitly reads as:

$$\bigcurlywedge_{a,a'} (a \to a') \to_j a \to_j a' \in S_j$$

i.e.:

$$j\left(\bigcurlywedge_{a,a'} (a \to a') \to j(a \to ja')\right) \in S$$

so, by (ii) in Definition 4.6, it suffices to show:

$$\bigcurlywedge_{a,a'} (a \to a') \to j(a \to ja') \in S$$

This, in turn, follows by intuitionistic reasoning from:

$$\bigcurlywedge_{a,a'} (a \to a') \to a \to ja' \in S$$

$$\bigcurlywedge_{a,a'} (a \to ja') \to j(a \to ja') \in S$$

which follow from (ii) Definition 4.6.

Then, let us show that j is an implicative morphism $\mathcal{A}_j \to \mathcal{A}$: again, recall that j is monotone by definition, so we only have to verify (i) and (ii) in Definition 5.1.

i. If $a \in S_j$, then by definition $ja \in S$.

ii. Condition (ii) explicitly reads as:

$$\bigcurlywedge_{a,a'} j(a \to ja') \to ja \to ja' \in S$$

which follows from intuitionistic reasoning from:

$$\bigcurlywedge_{a,a'} j(a \to ja') \to ja \to jja' \in S$$

$$\bigcurlywedge_{a'} jja' \to ja' \in S$$

Finally, let us show that $j : \mathcal{A}_j \to \mathcal{A}$ is right adjoint to $\mathrm{id}_A : \mathcal{A} \to \mathcal{A}_j$ in ArrAlg.

– On one hand, $j \vdash^{j}_{A} \mathrm{id}_A$ explicitly reads as $j \vdash_A j$, which is clearly true.
– On the other, $\mathrm{id}_A \vdash_A j$ is true as j is a nucleus.

Moreover, we also have that $\mathrm{id}_A \vdash^{j}_{A} j$ as it explicitly reads as $\mathrm{id}_A \vdash_A jj$, which makes id_A an implicative surjection by Proposition 6.17.

□

Corollary 6.21. *Every nucleus* j *on* $\mathcal{A}$ *induces a geometric inclusion of triposes* $P_{\mathcal{A}_j} \hookrightarrow P_{\mathcal{A}}$*, given by:*

$$P_{\mathcal{A}_j} \underset{j\circ -}{\overset{\mathrm{id}_A \circ -}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} P_{\mathcal{A}}$$

However, we are now in the position to do more than that: namely, we can recover Corollary 2.32 and hence a converse to the previous through nuclei.

Recall in fact by the discussion in Chapter 2 that we have an equivalence of preorder categories between subtriposes of $P_{\mathcal{A}}$ and closure transformations on $P_{\mathcal{A}}$, that is, transformations $\Phi_j : P_{\mathcal{A}} \to P_{\mathcal{A}}$ which are left exact, inflationary and idempotent:

$$\mathsf{SubTrip}(P_{\mathcal{A}}) \simeq \mathsf{ClTrans}(P_{\mathcal{A}})^{\mathrm{op}}$$

Note then that, given a closure transformation Φ_j on $P_{\mathcal{A}}$, the function $j \coloneqq (\Phi_j)_A(\mathrm{id}_A) : A \to A$ inducing it up to isomorphism satisfies the following:

i. j is an implicative morphism $\mathcal{A} \to \mathcal{A}$;
ii. $\mathrm{id}_A \vdash j$;
iii. $jj \dashv\vdash j$.

Assuming j to be monotone with respect to the evidential order in $\mathcal{A}$ as in Lemma 5.9, this means that j satisfies (i), (ii), (iv) and (vi) in Definition 4.6, which as we've noted suffice to make j into a nucleus. Of course, the converse is also true – that is, nuclei induce by postcomposition transformations which are left exact, inflationary and idempotent – since, as we know, every nucleus on $\mathcal{A}$ is an implicative morphism $\mathcal{A} \to \mathcal{A}$. Since the association $j \mapsto \Phi_j$ also preserves and reflects the order, we conclude with the following.

Proposition 6.22. *Let* $\mathsf{N}(\mathcal{A})$ *be the set of nuclei on* $\mathcal{A}$*, with the preorder induced by* $\mathrm{P}_{\mathcal{A}}(A)$*. Then, Proposition 6.4 yields an equivalence of preorder categories:*

$$\mathsf{ClTrans}(\mathrm{P}_{\mathcal{A}}) \simeq \mathsf{N}(\mathcal{A})$$

so, in particular:

$$\mathsf{SubTrip}(\mathrm{P}_{\mathcal{A}}) \simeq \mathsf{N}(\mathcal{A})^{\mathrm{op}}$$

Corollary 6.23. *Every geometric inclusion of toposes into* $\mathsf{AT}(\mathcal{A})$ *is induced, up to equivalence, by a geometric inclusion of triposes of the form:*

$$\mathrm{P}_{\mathcal{A}_j} \underset{j\circ -}{\overset{\mathrm{id}_A \circ -}{\underset{\bot}{\leftrightarrows}}} \mathrm{P}_{\mathcal{A}}$$

for some nucleus j *on* $\mathcal{A}$*.*

Remark 6.24. By definition of $\mathcal{A}_j$, note therefore how $\mathrm{P}_{\mathcal{A}_j}$ coincides precisely with the tripos P_j described before Corollary 2.32. For this reason, we will usually refer to $\mathrm{P}_{\mathcal{A}_j}$ simply as P_j.

We conclude this part with the following alternative description of P_j, already noted in the general case in [34] and then in the context of arrow algebras in [2].

Proposition 6.25. *Let* j *be a nucleus on an arrow algebra* $\mathcal{A}$*. Then,* P_j *is equivalent over* $\mathrm{P}_{\mathcal{A}}$ *to the tripos defined by:*

$$\mathrm{Q}_j(I) \coloneqq \{\, \alpha \in \mathrm{P}_{\mathcal{A}}(I) \mid j\alpha \vdash_I \alpha \,\}$$

with the Heyting prealgebra structure induced by $\mathrm{P}_{\mathcal{A}}(I)$*.*

Proof. Consider the pair of transformations:

$$\Theta^+ \coloneqq \mathrm{id}_A \circ - : \mathrm{Q}_j \to \mathrm{P}_j \qquad \Theta_+ \coloneqq j \circ - : \mathrm{P}_j \to \mathrm{Q}_j$$

obviously well-defined since so is the geometric morphism $\mathrm{P}_j \hookrightarrow \mathrm{P}_{\mathcal{A}}$ above, and as $jj \vdash_A j$. Then, Θ^+ and Θ_+ define an equivalence of triposes between P_j and Q_j.

- The fact that $\Theta^+\Theta_+ \simeq \mathrm{id}_{\mathrm{P}_j}$ is equivalent, by Proposition 6.4, to $j \dashv\vdash^j_A \mathrm{id}_A$: on one hand, $\mathrm{id}_A \vdash^j_A j$ explicitly means $\mathrm{id}_A \vdash_A jj$, which follows from $\mathrm{id}_A \vdash_A j$; on the other, $j \vdash^j_A \mathrm{id}_A$ explicitly means $j \vdash_A j$, which follows by reflexivity.

– To show that $\Theta_+\Theta^+ \simeq \mathrm{id}_{Q_j}$ we need to show that $j\alpha \dashv\vdash_I \alpha$ for every set I and every $\alpha \in Q_j(I)$: on one hand, $\alpha \vdash_I j\alpha$ follows by $\mathrm{id}_A \vdash_A j$; on the other, $j\alpha \vdash_I \alpha$ follows by definition of $Q_j(I)$.

Therefore, Q_j is equivalent to P_j; through the equivalence, the geometric inclusion of Q_j in $P_{\mathcal{A}}$ is given by:

$$Q_j \underset{j\circ -}{\overset{j\circ -}{\underset{\perp}{\leftrightarrows}}} P_{\mathcal{A}}$$

□

A FACTORIZATION THEOREM As it is known, every geometric morphism of toposes can be factored as a geometric surjection followed by a geometric inclusion. Generalizing locale theory, let us recover the same result in the framework of arrow algebras; in doing so, we will also make the correspondence between subtriposes and nuclei more explicit.

Let $f : \mathcal{A} \to \mathcal{B}$ be a computationally dense implicative morphism with right adjoint $h : \mathcal{B} \to \mathcal{A}$; by Lemma 5.9, up to isomorphism we can assume both f and h to be monotone with respect to the evidential order. First, observe the following.

Lemma 6.26. *$hf : \mathcal{A} \to \mathcal{A}$ is a nucleus on $\mathcal{A}$.*

Proof. Let us verify that hf satisfies the three conditions in Definition 4.6.

i. hf is monotone by composition.

ii. Clearly $\mathrm{id}_A \vdash hf$ as h is right adjoint to f.

iii. Let $I := A \times A$; note that condition (iii) can be rewritten as:

$$\pi_1 \to hf\pi_2 \vdash_I hf\pi_1 \to hf\pi_2$$

where $\pi_1, \pi_2 : I \to A$ are the obvious projections. Through the Heyting adjunction in $P_{\mathcal{A}}(I)$, this is equivalent to:

$$(\pi_1 \to hf\pi_2) \times hf\pi_1 \vdash_I hf\pi_2$$

and hence, since $h \circ -$ is right adjoint to $f \circ -$, which preserves finite meets, to:

$$f(\pi_1 \to hf\pi_2) \times fhf\pi_1 \vdash_I f\pi_2$$

Therefore, since $fhf\pi_1 \vdash_I f\pi_1$ as $fh \vdash \mathrm{id}_B$, it suffices to show:

$$f(\pi_1 \to hf\pi_2) \times f\pi_1 \vdash_I f\pi_2$$

i.e., again through the Heyting adjunction in $P_{\mathcal{B}}(I)$:

$$f(\pi_1 \to hf\pi_2) \vdash_I f\pi_1 \to f\pi_2$$

which in turn follows since, by (ii) in Definition 5.1:

$$f(\pi_1 \to hf\pi_2) \vdash_I f\pi_1 \to fhf\pi_2$$

and again $fhf\pi_2 \vdash_I f\pi_2$.

□

A natural question is then to relate f to $j := hf$, and in particular the geometric morphism $\Phi_f : P_{\mathcal{B}} \to P_{\mathcal{A}}$ induced by f with the inclusion $\Phi_j : P_j \hookrightarrow P_{\mathcal{A}}$ induced by j. To this aim, recall here that $fh \vdash id_B$ and $id_A \vdash hf$ imply $fhf \dashv\vdash f$ and $hfh \dashv\vdash h$.

Lemma 6.27. *f is an implicative injection $\mathcal{A}_j \to \mathcal{B}$, with h as a right adjoint.*

Proof. Let us start by showing that f is an implicative morphism $\mathcal{A}_j \to \mathcal{B}$.

i. Let $a \in S_j$; by definition, $hf(a) \in S_A$, so $fhf(a) \in S_B$, and hence $f(a) \in S_B$ since $fh \vdash id_B$.

ii. Condition (ii) explicitly reads as:

$$\bigwedge_{a,a'} f(a \to ja') \to f(a) \to f(a') \in S_B$$

which follows by intuitionistic reasoning from:

$$\bigwedge_{a,a'} f(a \to ja') \to f(a) \to fj(a') \in S_B$$

$$\bigwedge_{a,a'} (f(a) \to fhf(a')) \to f(a) \to f(a') \in S_B$$

where the latter follows since $fhf \vdash f$.

Note now that $h : B \to A$ is an implicative morphism $\mathcal{B} \to \mathcal{A}_j$ since it is an implicative morphism $\mathcal{B} \to \mathcal{A}$ and id_A is an implicative morphism $\mathcal{A} \to \mathcal{A}_j$. Then, we have that $h : \mathcal{B} \to \mathcal{A}_j$ is right adjoint to $f : \mathcal{A}_j \to \mathcal{B}$:

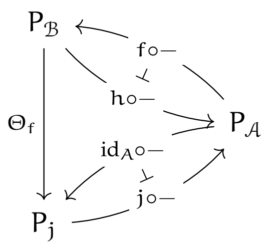
- clearly $fh \vdash id_B$;
- on the other hand, $id_A \vdash^j hf$ explicitly reads as $id_A \vdash_A hfhf$, which follows from $id_A \vdash_A hf$.

Moreover, we also have that $hf \vdash^j_A id_A$ as it explicitly reads as $hf \vdash_A hf$, which makes $f : \mathcal{A}_j \to \mathcal{B}$ an implicative surjection by Proposition 6.17. □

Recalling by Lemma 6.20 that id_A defines an implicative surjection $\mathcal{A} \to \mathcal{A}_j$, we have the following.

Corollary 6.28. *Every computationally dense implicative morphism factors as an implicative surjection followed by an implicative inclusion.*

On the level of triposes, this means that the geometric morphism $\Phi_f : \mathrm{P}_{\mathcal{B}} \to \mathrm{P}_{\mathcal{A}}$ induced by f factors through $\Phi_j : \mathrm{P}_j \hookrightarrow \mathrm{P}_{\mathcal{A}}$ by means of a geometric surjection $\Theta_f : \mathrm{P}_{\mathcal{B}} \to \mathrm{P}_j$, also induced by f as a morphism $\mathcal{A}_j \to \mathcal{B}$:



Proposition 6.29. *Θ_f is an equivalence if and only if Φ_f is an inclusion.*

Proof. By Lemma 6.12 and Proposition 6.13, Θ_f is an equivalence if and only if $f : \mathcal{A}_j \to \mathcal{B}$ is an equivalence. Since $h : \mathcal{B} \to \mathcal{A}_j$ is right adjoint to $f : \mathcal{A}_j \to \mathcal{B}$, this is equivalent to $hf \vdash^j_A \mathrm{id}_A$ and $\mathrm{id}_B \vdash fh$, and since $hf \vdash^j_A \mathrm{id}_A$ holds trivially this is equivalent simply to $\mathrm{id}_B \vdash fh$. By Proposition 6.17, $\mathrm{id}_B \vdash fh$ is in turn equivalent to Φ_f being an inclusion. □

Remark 6.30. In essence, this gives us a more explicit description of the correspondence given in Proposition 6.22, in perfect generalization of the localic case: indeed, if a subtripos $\Phi : \mathrm{P}_{\mathcal{B}} \hookrightarrow \mathrm{P}_{\mathcal{A}}$ is induced by an implicative surjection $f : \mathcal{A} \to \mathcal{B}$, then it is equivalent to the subtripos induced by (a nucleus isomorphic to) hf, where h is right adjoint to f.

6.4 EXAMPLES

We can finally conclude our analysis of the two main classes of arrow algebras, namely those arising from frames and from PCAs, now studying their morphisms in relation to the transformations between the associated triposes.

FRAMES First, recall that frame homomorphisms are implicative morphisms, seeing frames as arrow algebras in the canonical way. More generally, as noted in Remark 5.11, every monotone map of frames which preserves finite meets is an implicative morphisms: we can now easily prove the converse as well. In fact, if $f : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an implicative morphism, then:

- by definition, f is monotone with respect to the logical order;
- as the separator on a frame is canonically defined as $\{\top\}$, $f(\top) = \top$;
- by Lemma 6.2, f preserves binary logical meets.

Since the logical order on frames coincides with the evidential order, this simply means that f is monotone and preserves finite meets.

Moving on, let us see how every frame homomorphism is computationally dense as an implicative morphism.

Proposition 6.31. *Let* $f^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ *be a frame homomorphism. Then,* f^* *is a computationally dense implicative morphism.*

Proof. Let $f_* : \mathcal{O}(X) \to \mathcal{O}(Y)$ be the right adjoint of f^*, i.e. the monotone function:

$$f_*(x) = \bigvee \{ y \mid f^*(y) \leqslant x \}$$

As it is monotone and preserves finite meets, f_* is an implicative morphism $\mathcal{O}(X) \to \mathcal{O}(Y)$; in particular, it is clearly right adjoint to f^* in ArrAlg, which is then computationally dense.

□

The converse is also true: computationally dense implicative morphisms between frames are themselves frame homomorphisms. Indeed, let $f : \mathcal{O}(Y) \to \mathcal{O}(X)$ be a computationally dense implicative morphism between frames and let $h : \mathcal{O}(X) \to \mathcal{O}(Y)$ be right adjoint to it.

First, note that f is monotone, since the logical and the evidential order coincide on arrow algebras arising from frames and implicative morphisms are monotone with respect to the logical order. For the same reason, Lemma 6.2 implies that f preserves finite meets: indeed, for all $y \in \mathcal{O}(Y)$ we know that $\top \leqslant (f(y) \wedge f(y')) \to f(y \wedge y')$ i.e. $f(y) \wedge f(y') \leqslant f(y \wedge y')$, and therefore $f(y) \wedge f(y') = f(y \wedge y')$ as the converse inequality holds by monotonicity.

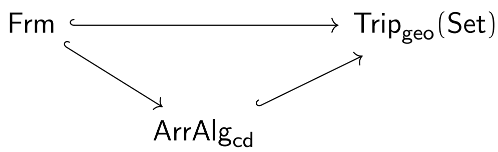
Finally, again as the logical and the evidential order coincide, h is then right adjoint to f as monotone maps between the posets underlying $\mathcal{O}(Y)$ and $\mathcal{O}(X)$, which means that f preserves all joins. Therefore, f is a morphism of frames; summing up, we have shown the following.

Proposition 6.32. *The inclusion 2-functor* $\mathsf{Frm} \hookrightarrow \mathsf{ArrAlg}_{\mathsf{cd}}$ *is 2-fully-faithful.*

Explicitly, this means that for any frames $\mathcal{O}(Y)$ *and* $\mathcal{O}(X)$ *there is an equivalence of preorder categories:*

$$\mathsf{Frm}(\mathcal{O}(Y), \mathcal{O}(X)) \simeq \mathsf{ArrAlg}_{\mathsf{cd}}(\mathcal{O}(Y), \mathcal{O}(X))$$

Remark 6.33. In essence, this makes so that the canonical embedding of locales and their homomorphisms into localic triposes and geometric morphisms factors through arrow algebras and computationally dense implicative morphisms. In the 'algebraic' notation we have been using, this gives the following diagram:



PARTIAL COMBINATORY ALGEBRAS The results of this chapter allow us to finally bridge the gap between (partial applicative) morphisms of PCAs and implicative morphisms of the associated arrow algebras.

First of all, recall by Lemma 5.12 that any morphism of PCAs $\mathrm{D}\,\mathbb{A} \to \mathrm{D}\,\mathbb{B}$, given two PCAs $\mathbb{A} = (A, \leqslant, \cdot, A^{\#})$ and $\mathbb{B} = (B, \leqslant, \cdot, B^{\#})$, is an implicative morphism between the associated arrow algebras. Proposition 6.4 now allows us to easily address the converse.

Proposition 6.34. *Let* $f : \mathrm{D}\,\mathbb{A} \to \mathrm{D}\,\mathbb{B}$ *be an implicative morphism. Then,* f *is also a morphism of PCAs* $\mathrm{D}\,\mathbb{A} \to \mathrm{D}\,\mathbb{B}$.

Therefore:

$$\mathsf{OPCA}(\mathrm{D}\,\mathbb{A}, \mathrm{D}\,\mathbb{B}) = \mathsf{ArrAlg}(\mathrm{D}\,\mathbb{A}, \mathrm{D}\,\mathbb{B})$$

Proof. Indeed, f induces by postcomposition the left exact transformation of realizability triposes $\Phi_f^+ : \mathrm{P}_{\mathrm{D}\,\mathbb{A}} \to \mathrm{P}_{\mathrm{D}\,\mathbb{B}}$; by Proposition 3.15, therefore, f is a morphism of PCAs $\mathrm{D}\,\mathbb{A} \to \mathrm{D}\,\mathbb{B}$. Recalling that the two orders coincide as well, we conclude that $\mathsf{OPCA}(\mathrm{D}\,\mathbb{A}, \mathrm{D}\,\mathbb{B}) = \mathsf{ArrAlg}(\mathrm{D}\,\mathbb{A}, \mathrm{D}\,\mathbb{B})$. $\square$

Moving on to partial applicative morphisms $\mathbb{A} \to \mathbb{B}$, recall that they correspond to regular transformations of triposes. The following definition is then obvious.

Definition 6.35. Let $\mathcal{A}$ and $\mathcal{B}$ be arrow algebras. An implicative morphism $f : \mathcal{A} \to \mathcal{B}$ is *regular* if:

$$f \circ \exists_g(\alpha) \dashv\vdash_Y \exists_g(f \circ \alpha)$$

for all functions $g : X \to Y$ and all $\alpha \in P_{\mathcal{A}}(X)$.

We denote with $\mathsf{ArrAlg}_{\mathsf{reg}}$ the wide subcategory of ArrAlg on regular implicative morphisms; the 2-functor of Proposition 6.4 obviously restricts to a 2-fully faithful 2-functor $\mathsf{ArrAlg}_{\mathsf{reg}} \to \mathsf{Trip}_{\mathsf{reg}}(\mathsf{Set})$.

Remark 6.36. Note that the inequality $\exists_g(f\alpha) \vdash_Y f\exists_g(\alpha)$ holds for all implicative morphisms f: indeed, through the adjunction $\exists_g \dashv g^*$ it is equivalent to $f\alpha \vdash_X f\exists_g(\alpha)g$, which is ensured by the properties of f since $\alpha \vdash_X \exists_g(\alpha)g$ by the unit of the same adjunction.

Therefore, regularity amounts to the inequality $f\exists_g(\alpha) \vdash_Y \exists_g(f\alpha)$.

Remark 6.37. Computationally dense implicative morphism are regular. Indirectly, this is obvious as inverse images of geometric morphisms of triposes are regular; more explicitly, instead, if $h : \mathcal{B} \to \mathcal{A}$ is right adjoint to f:

$$\begin{aligned} f\exists_g(\alpha) \vdash_Y \exists_g(f\alpha) &\iff \exists_g(\alpha) \vdash_Y h\exists_g(f\alpha) \\ &\iff \alpha \vdash_X h\exists_g(f\alpha)g \\ &\iff f\alpha \vdash_X \exists_g(f\alpha)g \end{aligned}$$

which is ensured by the unit of the adjunction $\exists_g \dashv g^*$.

Drawing from the previous results, we conclude that regular implicative morphisms between arrow algebras arising from PCAs arise themselves from partial applicative morphisms.

Proposition 6.38. *The 2-functor $\widetilde{D}$ of Proposition 5.14 restricts to a 2-fully faithful 2-functor:*

$$\mathsf{OPCA}_D \longrightarrow \mathsf{ArrAlg}_{\mathsf{reg}}$$

Explicitly, this means that for all PCAs $\mathbb{A}$ and $\mathbb{B}$, $\widetilde{D}$ realizes an equivalence of preorder categories:

$$\mathsf{OPCA}_D(\mathbb{A}, \mathbb{B}) \simeq \mathsf{ArrAlg}_{\mathsf{reg}}(D\,\mathbb{A}, D\,\mathbb{B})$$

Proof. Let $f : D\,\mathbb{A} \to D\,\mathbb{B}$ be a regular implicative morphism. Then, f induces by postcomposition the regular transformation of realizability triposes $\Phi_f^+ : P_{D\,\mathbb{A}} \to P_{D\,\mathbb{B}}$; by Proposition 3.18, therefore, $f = \widetilde{D}g$ for an essentially unique partial applicative morphism $g : \mathbb{A} \to \mathbb{B}$[5].

Moreover, for $f, f' : \mathbb{A} \to \mathbb{B}$ partial applicative morphisms, we have already showed that $f \leqslant f'$ in $\mathsf{OPCA_D}(\mathbb{A}, \mathbb{B})$ if and only if $\widetilde{D}f \vdash \widetilde{D}f'$ in $\mathsf{ArrAlg}(D\,\mathbb{A}, D\,\mathbb{B})$. □

Remark 6.39. Alternatively, the proof of the previous can be given by observing that a regular implicative morphism $f : D\,\mathbb{A} \to D\,\mathbb{B}$ is a union-preserving morphism of PCAs, and hence a D-algebra morphism: in fact, $D\,\mathbb{A}$ and $D\,\mathbb{B}$ are compatible with joins, meaning that existentials can be computed as unions.

Finally, let us specialize to the case of computational density. Recall by Lemma 3.19 that a partial applicative morphism $f : \mathbb{A} \to \mathbb{B}$ is computationally dense if and only if $\widetilde{f} : D\,\mathbb{A} \to D\,\mathbb{B}$ has a right adjoint in OPCA. Since $\mathsf{OPCA}(D\,\mathbb{A}, D\,\mathbb{B})$ coincides with $\mathsf{ArrAlg}(D\,\mathbb{A}, D\,\mathbb{B})$, this is also equivalent to $\widetilde{f} : D\,\mathbb{A} \to D\,\mathbb{B}$ having a right adjoint in ArrAlg, that is, to $\widetilde{f}$ being computationally dense as an implicative morphism $D\,\mathbb{A} \to D\,\mathbb{B}$. In essence, we have shown the following.

Proposition 6.40. *The 2-functor $\widetilde{D}$ of Proposition 5.14 restricts to a 2-fully faithful 2-functor:*

$$\mathsf{OPCA}_{D,cd} \longrightarrow \mathsf{ArrAlg}_{cd}$$

Explicitly, this means that for all PCAs $\mathbb{A}$ and $\mathbb{B}$, $\widetilde{D}$ realizes an equivalence of preorder categories:

$$\mathsf{OPCA}_{D\,cd}(\mathbb{A}, \mathbb{B}) \simeq \mathsf{ArrAlg}_{cd}(D\,\mathbb{A}, D\,\mathbb{B})$$

Remark 6.41. As for frames, in essence this makes so that the construction of realizability triposes and geometric morphisms from PCAs and partial applicative morphisms factors through arrow algebras and computationally dense implicative morphisms, giving the following diagram:

5 We can also describe g as $f \circ \delta'_{\mathbb{A}}$.

$$\begin{array}{ccc} \mathsf{OPCA}_{\mathrm{D}\,\mathsf{cd}} & \xhookrightarrow{\qquad\qquad} & \mathsf{Trip}_{\mathsf{geo}}(\mathsf{Set}) \\ & \searrow \quad \nearrow & \\ & \mathsf{ArrAlg}_{\mathsf{cd}} & \end{array}$$

7 ARROW ALGEBRAS FOR MODIFIED REALIZABILITY

In this chapter, we will apply the theoretical framework developed above to the study of *modified realizability* from the point of view of arrow algebras, recovering some functoriality results from [21] in greater generality.

The key feature of modified realizability lies in separating between a set of *potential realizers* and a subset thereof of *actual realizers*. On the level of triposes, this amounts to moving from the ordinary realizability tripos $P_{\mathbb{A}}$ over a (traditionally, discrete and absolute) PCA $\mathbb{A}$ to a tripos whose predicates on a set I are functions from I to the set:

$$\{ (\alpha, \beta) \in DA \times DA \mid \alpha \subseteq \beta \}$$

which are preordered by:

$$\phi \vdash_I \psi \iff \bigcap_{i \in I} (\phi_1(i) \to \psi_1(i)) \cap (\phi_2(i) \to \psi_2(i)) \in (DA)^{\#}$$

where we denote with $\phi_1(i), \phi_2(i)$ the two components of $\phi(i)$. This idea is what led, in [2], to the definition of the *Sierpiński construction* on arrow algebras, which we will describe below. To do this, and for what follows, we will consider some *ad hoc* notions of arrow algebras which still encompass all the relevant cases and many others. This does not mean that we have counterexamples showing how the presented results may fail for more general classes of arrow algebras, but only that minimal assumptions suffice to ensure the desired properties.

Definition 7.1. An arrow algebra $\mathcal{A} = (A, \preccurlyeq, \to, S)$ is *binary implicative* if the equality:

$$a \to (b \curlywedge c) = a \to b \curlywedge a \to c$$

holds for all $a, b, c \in A$, and it is *modifiable* if moreover the equality:

$$\bot \to a = \top$$

holds for all $a \in A$.

We denote with $\mathsf{ArrAlg}_{\mathsf{bi}}$ and $\mathsf{ArrAlg}_{\mathsf{mod}}$ the full subcategories of ArrAlg on binary implicative and modifiable arrow algebras, respectively.

Example 7.2. Every frame, seen as an arrow algebra in the canonical way, is modifiable.

Example 7.3. For any PCA $\mathbb{A}$, D $\mathbb{A}$ is modifiable; in particular, PER $\mathbb{A}$ is modifiable.

7.1 THE SIERPIŃSKI CONSTRUCTION

Recall by [2, Prop. 7.2] that, starting from any binary implicative arrow algebra $\mathcal{A} = (A, \preccurlyeq, \to, S)$, we can define a new arrow algebra $\mathcal{A}^{\to} = (A^{\to}, \preccurlyeq, \to, S^{\to})$, also binary implicative, by letting:

$$A^{\to} \coloneqq \{ x = (x_0, x_1) \in A \times A \mid x_0 \preccurlyeq x_1 \}$$

with pointwise order, implication:

$$x \to y \coloneqq (x_0 \to y_0 \curlywedge x_1 \to y_1, x_1 \to y_1)$$

and separator:

$$S^{\to} \coloneqq \{ x \in A^{\to} \mid x_0 \in S \}$$

Remark 7.4. This means that, for any set I, the order in $P_{\mathcal{A}^{\to}}(I)$ is given by:

$$\phi \vdash_I \psi \iff \bigcurlywedge_{i \in I} \phi_1(i) \to \psi_1(i) \curlywedge \phi_2(i) \to \phi_2(i) \in S$$

where we denote with $\phi_1, \phi_2 : I \to A$ the two components of $\phi : I \to A^{\to}$.

Let us now lift the association $\mathcal{A} \mapsto \mathcal{A}^{\to}$ to a (pseudo)functor on $\mathsf{ArrAlg}_{\mathsf{bi}}$.

Let $f : \mathcal{A} \to \mathcal{B}$ be an implicative morphism in $\mathsf{ArrAlg}_{\mathsf{bi}}$, for the moment assumed to be monotone, and define:

$$f^{\to} : A^{\to} \to B^{\to} \qquad f^{\to}(x_0, x_1) \coloneqq (f(x_0), f(x_1))$$

Lemma 7.5. *$f^{\to}$ is an implicative morphism $\mathcal{A}^{\to} \to \mathcal{B}^{\to}$.*

Proof. First, note that $f^{\to}$ is well-defined as a function $A^{\to} \to B^{\to}$ by monotonicity of f, and it is monotone itself with respect to the evidential orders in $\mathcal{A}^{\to}$ and $\mathcal{B}^{\to}$. Let us then verify that $f^{\to}$ satisfies the first two conditions in Definition 5.1.

i. If $x \in S_A^{\to}$, then $x_0 \in S_A$, so $f(x_0) \in S_B$ and hence $f^{\to}(x) \in S_B^{\to}$.

ii. First note that, for all $x, y \in A^{\to}$:

$$\begin{aligned} f^{\to}(x \to y) &= f^{\to}(x_0 \to y_0 \curlywedge x_1 \to y_1, x_1 \to y_1) \\ &= (f(x_0 \to y_0 \curlywedge x_1 \to y_1), f(x_1 \to y_1)) \end{aligned}$$

whereas:

$$\begin{aligned} f^{\to}(x) \to f^{\to}(y) &= (fx_0, fx_1) \to (fy_0, fy_1) \\ &= (fx_0 \to fy_0 \curlywedge fx_1 \to fy_1, fx_1 \to fy_1) \end{aligned}$$

Therefore, by binary implicativity, a realizer for $f^{\to}$ amounts to an element $r \in S_B$ such that:

$$\begin{aligned} r &\preccurlyeq f(x_0 \to y_0 \curlywedge x_1 \to y_1) \to fx_0 \to fy_0 \\ r &\preccurlyeq f(x_0 \to y_0 \curlywedge x_1 \to y_1) \to fx_1 \to fy_1 \\ r &\preccurlyeq f(x_1 \to y_1) \to fx_1 \to fy_1 \end{aligned}$$

for all $x, y \in A^{\to}$, in which case $(r, r) \in S_B^{\to}$ realizes $f^{\to}$. By monotonicity of f, note then that it suffices to show that:

$$\begin{aligned} r &\preccurlyeq f(x_0 \to y_0) \to fx_0 \to fy_0 \\ r &\preccurlyeq f(x_1 \to y_1) \to fx_1 \to fy_1 \end{aligned}$$

for all $x, y \in A^{\to}$, which means that r can be taken to be a realizer for f.

□

Therefore, $(-)^{\to}$ defines a functorial association on binary implicative arrow algebras and monotone implicative morphisms between them. Note moreover that, given two monotone implicative morphisms $f, f' : \mathcal{A} \to \mathcal{B}$ in $\mathsf{ArrAlg}_{\mathsf{bi}}$, if $u \in S_B$ realizes $f \vdash f'$, then $(u, u) \in S_B^{\to}$ clearly realizes $f^{\to} \vdash f'^{\to}$, meaning that $(-)^{\to}$ is actually 2-functorial. Precomposing with the pseudofunctor M of Proposition 5.10, we obtain the following.

Proposition 7.6. *For any implicative morphism* $f : \mathcal{A} \to \mathcal{B}$ *in* $\mathsf{ArrAlg}_{\mathsf{bi}}$*, let:*

$$f^{\to} : \mathcal{A}^{\to} \to \mathcal{B}^{\to} \qquad f^{\to}(x_0, x_1) := \left(\bigcurlywedge_{x_0 \preccurlyeq a} \partial f(a), \bigcurlywedge_{x_1 \preccurlyeq a} \partial f(a) \right)$$

Then, $(-)^{\to}$ *is a pseudofunctor* $\mathsf{ArrAlg}_{\mathsf{bi}} \to \mathsf{ArrAlg}_{\mathsf{bi}}$.

Moreover, if f *is computationally dense with right adjoint* $h : \mathcal{B} \to \mathcal{A}$*, then* $f^{\to}$ *is computationally dense as well, and a right adjoint is given by* $h^{\to} : \mathcal{B}^{\to} \to \mathcal{A}^{\to}$.

Proof. We only have to show the last part, so let f be computationally dense with right adjoint $h : \mathcal{B} \to \mathcal{A}$ and, up to isomorphism, assume both f and h to be monotone, so that $f^{\to}$ and $h^{\to}$ can be defined as above in the case of monotonicity. Let's show that $h^{\to}$ is right adjoint to $f^{\to}$.

– To show that $f^{\to}h^{\to} \vdash id_{B^{\to}}$, note that:

$$\begin{aligned} & \curlywedge_{y \in B^{\to}} f^{\to}h^{\to}(y) \to y \in S_B^{\to} \\ \iff & \curlywedge_{y \in B^{\to}} (fh(y_0), fh(y_1)) \to (y_0, y_1) \in S_B^{\to} \\ \iff & \curlywedge_{y \in B^{\to}} fh(y_0) \to y_0 \curlywedge fh(y_1) \to y_1 \in S_B \end{aligned}$$

which is ensured by $fh \vdash id_B$.

– Similarly, $id_{A^{\to}} \vdash h^{\to}f^{\to}$ reduces to $id_A \vdash hf$.

□

Corollary 7.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be binary implicative arrow algebras.*

Then, every geometric morphism $\Phi : P_{\mathcal{B}} \to P_{\mathcal{A}}$ lifts to a geometric morphism $\Phi^{\to} : P_{\mathcal{B}^{\to}} \to P_{\mathcal{A}^{\to}}$.

Question. For $\mathcal{A} = \text{Pow}(\mathcal{K}_1)$, we have that $\mathsf{AT}(\mathcal{A}^{\to})$ is the effective topos built on the topos of sheaves over the Sierpiński space, $\mathsf{Eff}_{\cdot\to\cdot}$; that is, the result of the construction of Eff inside the topos $\mathsf{Set}^{\cdot\to\cdot}$.

Can we develop a theory of arrow algebras over other base toposes, encompassing that of PCAs over other base toposes, so that the same result holds for every (binary implicative) arrow algebra?

7.2 THE MODIFICATION OF AN ARROW ALGEBRA

Let us now study the relation between $P_{\mathcal{A}}$ and $P_{\mathcal{A}^{\to}}$. First, generalizing what is showed in [20] for discrete and absolute PCAs, we can note the following.

Lemma 7.8. *$P_{\mathcal{A}}$ is a subtripos of $P_{\mathcal{A}^{\to}}$.*

Proof. Consider the projection:

$$\pi_1 : A^{\to} \to A \qquad (x_0, x_1) \mapsto x_1$$

Let us show that π_1, which is obviously monotone, is an implicative morphism $\mathcal{A}^{\to} \to \mathcal{A}$.

i. If $(x_0, x_1) \in S^{\to}$, then by definition $x_0 \in S$, so $x_1 \in S$ as well since $x_0 \preccurlyeq x_1$.

ii. A realizer of π_1 amounts to an element $r \in S$ such that:

$$r \preccurlyeq (x_1 \to y_1) \to x_1 \to y_1$$

for all $x, y \in A^{\to}$, so we can take $r := \mathbf{i}$.

Consider now the diagonal map:

$$\delta : A \to A^{\to} \qquad a \mapsto (a, a)$$

Let us show that δ, also obviously monotone, is an implicative morphism $\mathcal{A} \to \mathcal{A}^{\to}$.

i. If $a \in S$, then clearly $(a, a) \in S$.

ii. We have:

$$\begin{aligned} &\bigcurlywedge_{a,a' \in A} \delta(a \to a') \to \delta(a) \to \delta(a') \in S^{\to} \\ \iff &\bigcurlywedge_{a,a' \in A} (a \to a', a \to a') \to (a, a) \to (a', a') \in S^{\to} \\ \iff &\bigcurlywedge_{a,a' \in A} (a \to a') \to a \to a' \in S \end{aligned}$$

which is ensured by $\mathbf{i} \in S$.

Finally, let us show that δ is right adjoint to π_1 in ArrAlg, making it an implicative surjection.

– On one hand, $\pi_1 \delta = \mathrm{id}_A$.
– On the other, we have:

$$\begin{aligned} &\mathrm{id}_{A^{\to}} \vdash_{A^{\to}} \delta\pi_0 \\ \iff &\bigcurlywedge_{x \in A^{\to}} (x_0, x_1) \to (x_1, x_1) \in S^{\to} \\ \iff &\bigcurlywedge_{x \in A^{\to}} x_1 \to x_1 \in S \end{aligned}$$

which is ensured by $\mathbf{i} \in S$.

Therefore, π_1 induces a geometric inclusion $\Phi_1 : \mathsf{P}_{\mathcal{A}} \hookrightarrow \mathsf{P}_{\mathcal{A}^{\to}}$. □

Corollary 7.9. *$\mathsf{AT}(\mathcal{A})$ is a subtopos of $\mathsf{AT}(\mathcal{A}^{\to})$.*

Question. In the case of $\mathcal{A} = \mathrm{Pow}(\mathbb{P})$ for a discrete and absolute PCA, Johnstone [20, Lem. 3.1] showed that there is another inclusion $\mathsf{P}_{\mathcal{A}} \hookrightarrow \mathsf{P}_{\mathcal{A}^{\to}}$, induced by the projection $\pi_0 : \mathcal{A}^{\to} \to \mathcal{A}$ and disjoint from Φ_1. We have not been able to show that this holds in general for (binary implicative) arrow algebras, nor to find reasonable assumptions under which this may be the case.

At least in the modifiable case, we can say even more about the inclusion $\Phi_1 : P_{\mathcal{A}} \hookrightarrow P_{\mathcal{A}^{\rightarrow}}$.

Specializing Definition 2.34 to the context of arrow algebras, recall that a subtripos of $P_{\mathcal{A}}$ is *open* if it is induced by a nucleus o on $\mathcal{A}$ of the shape:

$$o(a) \coloneqq u \to a$$

for some $u \in A$, in which case the *closed* nucleus:

$$c(a) \coloneqq a + u$$

induces its complement in the lattice of subtriposes of $P_{\mathcal{A}}$ considered up to equivalence.

Definition 7.10. Given a modifiable arrow algebra $\mathcal{A}$, we define its *modification* as the arrow algebra $\mathcal{A}^m \coloneqq (\mathcal{A}^{\rightarrow})_c$, where c is the nucleus on $\mathcal{A}^{\rightarrow}$ defined by:

$$c(x) \coloneqq x + (\bot, \top)$$

We denote with $M_{\mathcal{A}}$ the *modified arrow tripos* $P_{\mathcal{A}^m}$, that is, the subtripos $P_{(\mathcal{A}^{\rightarrow})_c}$ of $P_{\mathcal{A}^{\rightarrow}}$.

Proposition 7.11. *Let $\mathcal{A}$ be a modifiable arrow algebra. Then, the inclusion $\Phi_1 : P_{\mathcal{A}} \hookrightarrow P_{\mathcal{A}^{\rightarrow}}$ is open, induced by the nucleus:*

$$o(x) \coloneqq (\bot, \top) \to x$$

In particular, $M_{\mathcal{A}}$ is the closed complement of $P_{\mathcal{A}}$ in the lattice of subtriposes of $P_{\mathcal{A}^{\rightarrow}}$ considered up to equivalence.

Proof. By Remark 6.30 and the discussion preceding it, we only have to show that $o \dashv\vdash \delta\pi_1$.

– To show that $o \vdash_{A^{\rightarrow}} \delta\pi_1$, note that:

$$\begin{aligned} & \bigcurlywedge_{x \in A^{\rightarrow}} o(x) \rightarrow \delta\pi_1(x) \in S^{\rightarrow} \\ \iff & \bigcurlywedge_{x \in A^{\rightarrow}} ((\bot, \top) \rightarrow (x_0, x_1)) \rightarrow (x_1, x_1) \in S^{\rightarrow} \\ \iff & \bigcurlywedge_{x \in A^{\rightarrow}} (\bot \rightarrow x_0 \curlywedge \top \rightarrow x_1, \top \rightarrow x_1) \rightarrow (x_1, x_1) \in S^{\rightarrow} \\ \iff & \bigcurlywedge_{x \in A^{\rightarrow}} (\bot \rightarrow x_0 \curlywedge \top \rightarrow x_1) \rightarrow x_1 \curlywedge (\top \rightarrow x_1) \rightarrow x_1 \in S \\ \iff & \bigcurlywedge_{x \in A^{\rightarrow}} (\top \rightarrow x_1) \rightarrow x_1 \in S \end{aligned}$$

which is ensured by the properties of $\partial a \coloneqq \top \rightarrow a$.

– To show that $\delta\pi_1 \vdash_{A^{\rightarrow}} o$ note that, by the hypothesis of modifiability:

$$\begin{aligned} & \bigcurlywedge_{x \in A^{\rightarrow}} \delta\pi_1(x) \rightarrow o(x) \in S^{\rightarrow} \\ \iff & \bigcurlywedge_{x \in A^{\rightarrow}} (x_1, x_1) \rightarrow (\bot, \top) \rightarrow (x_0, x_1) \in S^{\rightarrow} \\ \iff & \bigcurlywedge_{x \in A^{\rightarrow}} (x_1, x_1) \rightarrow (\bot \rightarrow x_0 \curlywedge \top \rightarrow x_1, \top \rightarrow x_1) \in S^{\rightarrow} \\ \iff & \bigcurlywedge_{x \in A^{\rightarrow}} (x_1, x_1) \rightarrow (\top \rightarrow x_1, \top \rightarrow x_1) \in S^{\rightarrow} \\ \iff & \bigcurlywedge_{x \in A^{\rightarrow}} (x_1 \rightarrow \top \rightarrow x_1, x_1 \rightarrow \top \rightarrow x_1) \in S^{\rightarrow} \\ \iff & \bigcurlywedge_{x \in A^{\rightarrow}} x_1 \rightarrow \top \rightarrow x_1 \in S \end{aligned}$$

which is again ensured by the properties of $\partial a \coloneqq \top \rightarrow a$.

□

Example 7.12. For $\mathcal{A} = \mathrm{Pow}(\mathcal{K}_1)$, we reobtain what proved in [32]: the effective topos $\mathsf{Eff} \simeq \mathsf{AT}(\mathcal{A})$ is an open subtopos of the effective topos built on the topos of sheaves over the Sierpiński space, $\mathsf{Eff}_{\cdot \rightarrow \cdot} \simeq \mathsf{AT}(\mathcal{A}^{\rightarrow})$, and Grayson's modified realizability topos Mod – characterized in [2] as $\mathsf{AT}(\mathcal{A}^m)$ – is its closed complement.

Example 7.13. For $\mathcal{A} = \mathrm{PER}\,\mathbb{N}$, we obtain that the *extensional modified realizability topos* characterized in [2] as $\mathsf{AT}(\mathcal{A}^m)$ is the closed complement of $\mathsf{AT}(\mathcal{A})$ as subtoposes of $\mathsf{AT}(\mathcal{A}^{\rightarrow})$.

Let us now see how the construction of the modified arrow tripos can be made (pseudo)functorial. In the proof, we will need the following property, which makes use of the hypothesis of modifiability.

Lemma 7.14. *Let* $f : \mathcal{A} \to \mathcal{B}$ *be an implicative morphism in* $\mathsf{ArrAlg}_{\mathsf{mod}}$.
Then, $cf^{\to}c \vdash f^{\to}c$.[1]

Proof. By definition of the nucleus $c \in \mathsf{N}(\mathcal{B}^{\to})$, and using the fact that logical joins are computed pointwise in $(\mathcal{B}^{\to})^{A^{\to}}$, $cf^{\to}c \vdash f^{\to}c$ is equivalent to:

$$\bigwedge_{x \in A^{\to}} (\bot, \top) \to f^{\to}c(x) \in S_B^{\to}$$

Since $\mathcal{B}$ is modifiable, this reduces to:

$$\bigwedge_{x \in A^{\to}} \top \to Mf((cx)_1) \in S_B^{\to}$$

where $M : \mathsf{ArrAlg} \to \mathsf{ArrAlg}$ is the monotonization pseudofunctor of Proposition 5.10, and hence since $Mf \dashv\vdash f$:

$$\bigwedge_{x \in A^{\to}} \top \to f(((\bot, \top) + (x_0, x_1))_1) \in S_B^{\to}$$

Note now that, in any arrow algebra of the form $\mathcal{A}^{\to}$, the logical join $a + a'$ has $a_1 + a'_1$ as its second component. This can be seen using the explicit description of logical joins given in Proposition 4.20, recalling that (evidential) meets and implications in $\mathcal{A}^{\to}$ are computed pointwise on the second component. Therefore, the previous is equivalent to:

$$\bigwedge_{x \in A^{\to}} \top \to f(\top + x_1) \in S_B$$

which follows from $\bigwedge_a \top \to (\top + a) \in S_A$ by the properties of f. $\square$

Theorem 7.15. *For any implicative morphism* $f : \mathcal{A} \to \mathcal{B}$ *in* $\mathsf{ArrAlg}_{\mathsf{mod}}$, *let* $f^m : \mathcal{A}^m \to \mathcal{B}^m$ *be the composite:*

$$\mathcal{A}^m \xrightarrow{c} \mathcal{A}^{\to} \xrightarrow{f^{\to}} \mathcal{B}^{\to} \xrightarrow{\mathrm{id}_{B^{\to}}} \mathcal{B}^m$$

where $f^{\to} : \mathcal{A}^{\to} \to \mathcal{B}^{\to}$ *is the implicative morphism defined in Proposition 7.6.*
Then, $(-)^m$ *is a pseudofunctor* $\mathsf{ArrAlg}_{\mathsf{mod}} \to \mathsf{ArrAlg}$.

1 Of course, the first c is a nucleus on $\mathcal{A}^{\to}$, while the second one is a nucleus on $\mathcal{B}^{\to}$.

Moreover, if f *is computationally dense with right adjoint* $h : \mathcal{B} \to \mathcal{A}$*, then* f^m *is computationally dense as well, and a right adjoint is given by* $h^m : \mathcal{B}^m \to \mathcal{A}^m$*. Furthermore, the square:*

$$\begin{array}{ccc} \mathcal{B}^m & \xrightarrow{h^m} & \mathcal{A}^m \\ c\big\downarrow & & \big\downarrow c \\ \mathcal{B}^\to & \xrightarrow[h^\to]{} & \mathcal{A}^\to \end{array}$$

commutes up to isomorphism.

Proof. First, let us show that $(-)^m$ preserves identities and compositions up to isomorphism.

- By definition, $id^m_A : \mathcal{A}^m \to \mathcal{A}^m$ is given by the composite:

$$\mathcal{A}^m \xrightarrow{c} \mathcal{A}^\to \xrightarrow{id_A{}^\to} \mathcal{A}^\to \xrightarrow{id_{A^\to}} \mathcal{A}^m$$

  which means that $id^m_A = c : \mathcal{A}^m \to \mathcal{A}^m$, and obviously $c \dashv\vdash^c id_{A^\to}$.
- By definition, for $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$, $(gf)^m$ is given by the composite:

$$\mathcal{A}^m \xrightarrow{c} \mathcal{A}^\to \xrightarrow{(gf)^\to} \mathcal{C}^\to \xrightarrow{id_{C^\to}} \mathcal{C}^m$$

  where of course $(gf)^\to \dashv\vdash g^\to f^\to$, whereas $g^m f^m$ is given by the composite:

$$\mathcal{A}^m \xrightarrow{c} \mathcal{A}^\to \xrightarrow{f^\to} \mathcal{B}^\to \xrightarrow{id_{B^\to}} \mathcal{B}^m \xrightarrow{c} \mathcal{B}^\to \xrightarrow{g^\to} \mathcal{C}^\to \xrightarrow{id_{C^\to}} \mathcal{C}^m$$

  which means that we need to show that $g^\to c f^\to c \dashv\vdash^c g^\to f^\to c$.

  On one hand, using the fact that $id \vdash c$ both for $c \in N(\mathcal{B})$ and $c \in N(\mathcal{C})$:

$$g^\to f^\to c \vdash g^\to c f^\to c \vdash c g^\to c f^\to c$$

  i.e. $g^\to f^\to c \vdash^c g^\to c f^\to c$.

  On the other, by the previous lemma we know that $c f^\to c \vdash f^\to c$, which implies $g^\to c f^\to c \vdash g^\to f^\to c$ by the properties of $g^\to$, and hence $g^\to c f^\to c \vdash c g^\to f^\to c$ since $id_{C^\to} \vdash c$.

The pseudofunctoriality of $f \mapsto f^\to$ then yields the pseudofunctoriality of $f \mapsto f^m$.

Suppose now $h : \mathcal{B} \to \mathcal{A}$ is right adjoint to f and, without loss of generality, assume h to be monotone, in which case we can describe $h^\to$

as $(y_0, y_1) \mapsto (h(y_0), h(y_1))$; let us show that $h^{\to}c$ is right adjoint to $f^{\to}c : \mathcal{A}^m \to \mathcal{B}^m$.

– On one hand, $f^{\to}ch^{\to}c \vdash^{c}_{B^{\to}} \mathrm{id}_{B^{\to}}$ explicitly reads as $f^{\to}ch^{\to}c \vdash_{B^{\to}} c$. By Proposition 7.6, we know that $h^{\to}$ is right adjoint to $f^{\to}$, so the previous is equivalent to $ch^{\to}c \vdash_{B^{\to}} h^{\to}c$, which is ensured by the previous lemma.

– On the other, $\mathrm{id}_{A^{\to}} \vdash^{c}_{A^{\to}} h^{\to}cf^{\to}c$ explicitly reads as $\mathrm{id}_{A^{\to}} \vdash_{A^{\to}} ch^{\to}cf^{\to}c$. As $\mathrm{id}_{A^{\to}} \vdash c$, this is ensured if $\mathrm{id}_{A^{\to}} \vdash_{A^{\to}} h^{\to}cf^{\to}c$. This is again equivalent to $f^{\to} \vdash_{A^{\to}} cf^{\to}c$ as $h^{\to}$ is right adjoint to $f^{\to}$, which follows since $\mathrm{id} \vdash c$, both for $c \in N(\mathcal{A}^{\to})$ and $c \in N(\mathcal{B}^{\to})$.

Finally, to show that the square above commutes up to isomorphism, we need to show that $ch^m \dashv\vdash h^{\to}c$ as morphisms $\mathcal{B}^m \to \mathcal{A}^{\to}$. On one hand, $h^{\to}c \vdash ch^m$ explicitly means $h^{\to}c \vdash_{B^{\to}} ch^{\to}c$, which follows simply being $\mathrm{id}_{A^{\to}} \vdash c$. On the other, $ch^m \vdash h^{\to}c$ explicitly means $ch^{\to}c \vdash_{B^{\to}} h^{\to}c$, which is again ensured by the previous lemma. □

Corollary 7.16. *Let $\mathcal{A}$ and $\mathcal{B}$ be modifiable arrow algebras. Then, every geometric morphism $\Phi : P_{\mathcal{B}} \to P_{\mathcal{A}}$ induces a geometric morphism $\Phi^m : M_{\mathcal{B}} \to M_{\mathcal{A}}$ such that the diagram:*

$$\begin{array}{ccc} M_{\mathcal{B}} & \xrightarrow{\Phi^m} & M_{\mathcal{A}} \\ \downarrow & & \downarrow \\ P_{\mathcal{B}^{\to}} & \xrightarrow[\Phi^{\to}]{} & P_{\mathcal{A}^{\to}} \end{array}$$

is a pullback square of triposes and geometric morphisms.

In particular, the induced diagram of toposes and geometric morphisms:

$$\begin{array}{ccc} \mathsf{AT}(\mathcal{B}^m) & \longrightarrow & \mathsf{AT}(\mathcal{A}^m) \\ \downarrow & & \downarrow \\ \mathsf{AT}(\mathcal{B}^{\to}) & \longrightarrow & \mathsf{AT}(\mathcal{A}^{\to}) \end{array}$$

is a pullback square.

Proof. The fact that the square commutes follows directly by the previous proposition. To show that it is a pullback, instead, recall from [21] that, given a closed nucleus $kx := x + u$ on $\mathcal{A}^{\to}$, the pullback of the closed subtripos $P_{\mathcal{A}^{\to}_k} \hookrightarrow P_{\mathcal{A}^{\to}}$ along $\Phi^{\to}$ is the closed subtripos of $P_{\mathcal{B}^{\to}}$ determined by the nucleus $k'y := y + (\Phi^{\to})^{+}_{B^{\to}}(u)$. Therefore, the square above is a pullback if and only if $(\Phi^{\to})^{+}_{B^{\to}}(\perp, \top) \dashv\vdash (\perp, \top)$ in $\mathcal{B}^{\to}$.

To prove this, let $f : \mathcal{A} \to \mathcal{B}$ be an implicative morphism with right adjoint $h : \mathcal{B} \to \mathcal{A}$ inducing Φ, so that $\Phi^{\to}$ is induced by $f^{\to}$ with right adjoint $h^{\to}$ as in Proposition 7.6; then, we need to show that $f^{\to}(\bot, \top) \dashv\vdash (\bot, \top)$ in $\mathcal{B}^{\to}$. On one hand, by modifiability of $\mathcal{B}$, $(\bot, \top) \vdash f^{\to}(\bot, \top)$ reduces simply to $\top \vdash f(\top)$, which is true as f is an implicative morphism. On the other, $f^{\to}(\bot, \top) \vdash (\bot, \top)$ is equivalent to $(\bot, \top) \vdash h^{\to}(\bot, \top)$, which is true again as $\mathcal{A}$ is modifiable and h is an implicative morphism. □

Remark 7.17. In particular, restricting to arrow algebras of the form $\mathrm{Pow}(\mathbb{P})$ for a discrete and absolute PCA $\mathbb{P}$, we reobtain [21, Prop. 2.1].

Remark 7.18. Recall by Proposition 6.25 that we can identify $M_{\mathcal{A}}$ up to equivalence with the subtripos $M'_{\mathcal{A}} \hookrightarrow P_{\mathcal{A}^{\to}}$ defined by:

$$\begin{aligned} M'_{\mathcal{A}}(I) &:= \{\, \alpha \in P_{\mathcal{A}^{\to}}(I) \mid c\alpha \vdash_I \alpha \,\} \\ &= \left\{ \alpha \in P_{\mathcal{A}^{\to}}(I) \;\middle|\; \bigcurlywedge_i (\bot, \top) \to \alpha(i) \in S^{\to}_{A} \right\} \\ &= \left\{ \alpha \in P_{\mathcal{A}^{\to}}(I) \;\middle|\; \bigcurlywedge_i \top \to \alpha_1(i) \in S_{A} \right\} \\ &= \{\, \alpha \in P_{\mathcal{A}^{\to}}(I) \mid \top_I \vdash_I \alpha_1 \,\} \end{aligned}$$

and in the same way we can identify $M_{\mathcal{B}}$ up to equivalence with the subtripos $M'_{\mathcal{B}} \hookrightarrow P_{\mathcal{B}^{\to}}$ defined by:

$$M'_{\mathcal{B}}(I) = \{\, \beta \in P_{\mathcal{B}^{\to}}(I) \mid \top_I \vdash_I \beta_1 \,\}$$

In these terms, Φ^m can be described explicitly as:

$$M'_{\mathcal{B}} \underset{h^{\to}\circ-}{\overset{f^{\to}\circ-}{\rightleftarrows}} M'_{\mathcal{A}} \quad (\bot)$$

that is, exactly the restriction of $\Phi^{\to}$ in both directions.

The details of the proof of the previous corollary also reveal that a similar result holds for open complements of modified triposes, again generalizing what proved in [21].

Proposition 7.19. *Let $\mathcal{A}$ and $\mathcal{B}$ be modifiable arrow algebras. Then, for every geometric morphism $\Phi : \mathsf{P}_{\mathcal{B}} \to \mathsf{P}_{\mathcal{A}}$, the diagram:*

$$\begin{array}{ccc} \mathsf{P}_{\mathcal{B}} & \xrightarrow{\Phi} & \mathsf{P}_{\mathcal{A}} \\ \big\downarrow & & \big\downarrow \\ \mathsf{P}_{\mathcal{B}^{\to}} & \xrightarrow[\Phi^{\to}]{} & \mathsf{P}_{\mathcal{A}^{\to}} \end{array}$$

is a pullback square of triposes and geometric morphisms.

In particular, the induced diagram of toposes and geometric morphisms:

$$\begin{array}{ccc} \mathsf{AT}(\mathcal{B}) & \longrightarrow & \mathsf{AT}(\mathcal{A}) \\ \big\downarrow & & \big\downarrow \\ \mathsf{AT}(\mathcal{B}^{\to}) & \longrightarrow & \mathsf{AT}(\mathcal{A}^{\to}) \end{array}$$

is a pullback square.

8

ARROW ASSEMBLIES

We conclude this thesis with the first steps towards the introduction of a suitable category of *assemblies* for arrow algebras, generalizing the notion of assemblies for a PCA.

A notion of assemblies in the case of relative ordered partial combinatory algebras is given by [39], where it is shown how assemblies sits inside the corresponding realizability topos as a full subcategory whose ex/reg completion coincides (up to equivalence) with the realizability topos itself. Instead, our definition will follows that of [7] for implicative algebras, which does not directly generalize Zoethout's definition but rather the traditional notion for the discrete and absolute case.

The category of assemblies over a PCA, in either definition, is a quasitopos with a natural number object which, in many relevant cases, allows us to simplify the study of the internal logic of the corresponding realizability topos – see, for example, the use of assemblies in [38] for the study of a topos for modified realizability. The goal, in this sense, would be to build a sufficiently strong theory of assemblies over an arrow algebra in order to perform a similar reduction.

8.1 THE CATEGORY OF ARROW ASSEMBLIES

Let $\mathcal{A} = (A, \preccurlyeq, \to, S)$ be an arrow algebra and let $P : \mathsf{Set}^{op} \to \mathsf{HeytPre}$ be the associated arrow tripos. The following definition is given in [7] for implicative algebras, but it works in the setting of arrow algebras as well.

Definition 8.1. We define the category $\mathsf{ArrAsm}(\mathcal{A})$ of *arrow assemblies* as the category having:

- as objects, pairs (X, α) of a set X and a function $\alpha : X \to S$.

– as morphisms $(X, \alpha) \to (Y, \beta)$, functions $f : X \to Y$ satisfying the *tracking condition* $\alpha \vdash_X f^*(\beta)$, i.e. explicitly:

$$\bigcurlywedge_{x \in X} \alpha(x) \to \beta(f(x)) \in S$$

with compositions and identities defined as in Set.

Given an arrow assembly (X, α), we say that X is the *carrier*, and α is the *existence predicate*. We denote with Γ the obvious forgetful functor $\mathsf{ArrAsm}(\mathcal{A}) \to \mathsf{Set}$.

Example 8.2. We recover the usual definition of assemblies for a PCA in the case of $\mathcal{A} = \mathrm{Pow}(\mathbb{P})$.

Let us now prove that, as in the case of PCAs and implicative algebras, $\mathsf{ArrAsm}(\mathcal{A})$ is a regular category.

Lemma 8.3. $\mathsf{ArrAsm}(\mathcal{A})$ *is finitely complete.*

Proof. The terminal object is clearly given by $(\{*\}, \top)$.

The product of (X, α) and (Y, β) is given by $(X \times Y, \alpha \otimes \beta)$, where:

$$\alpha \otimes \beta := \pi_X^*(\alpha) \times \pi_Y^*(\beta) : X \times Y \to S \qquad (x, y) \mapsto \alpha(x) \times \beta(y)$$

together with the two projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$, which are trivially tracked as morphisms $(X \times Y, \alpha \otimes \beta) \to (X, \alpha)$ and $(X \times Y, \alpha \otimes \beta) \to (Y, \beta)$.

The equalizer of two morphisms $f, g : (X, \alpha) \to (Y, \beta)$ is given by $(E, i^*(\alpha))$, where $E := \{ x \in X \mid f(x) = g(x) \}$ is the equalizer of f and g in Set and $i : E \hookrightarrow X$ is the inclusion, which is trivially tracked as a morphism $(E, \alpha) \to (X, \alpha)$. □

Corollary 8.4. $\Gamma : \mathsf{ArrAsm}(\mathcal{A}) \to \mathsf{Set}$ *preserves finite limits.*

Remark 8.5. By the construction of products and equalizer we can also describe pullbacks explicitly. Indeed, the pullback of $f : (X, \alpha) \to (Z, \gamma)$ and $g : (Y, \beta) \to (Z, \gamma)$ is given by:

$$\begin{array}{ccc} (P, p_0^*(\alpha) \times p_1^*(\beta)) & \xrightarrow{p_0} & (X, \alpha) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ (Y, \beta) & \xrightarrow[g]{} & (Z, \gamma) \end{array}$$

where $P := \{ (x, y) \in X \times Y \mid f(x) = g(y) \}$ is the pullback of f and g in Set and $p_0 : P \to X$ and $p_1 : P \to Y$ are the two projections.

Lemma 8.6. $\mathsf{ArrAsm}(\mathcal{A})$ *is finitely cocomplete.*

Proof. The initial object is clearly given by $(\emptyset, \emptyset)$.

The coproduct of (X, α) and (Y, β) is given by $(X \sqcup Y, \alpha \oplus \beta)$, where:

$$\alpha \oplus \beta := \exists_{\iota_X}(\alpha) + \exists_{\iota_Y}(\beta) : X \sqcup Y \to S$$

together with the two coprojections $\iota_X : X \to X \sqcup Y$ and $\iota_Y : Y \to X \sqcup Y$, which are tracked as morphisms $(X, \alpha) \to (X \sqcup Y, \alpha \oplus \beta)$ and $(Y, \beta) \to (X \sqcup Y, \alpha \oplus \beta)$ since $\alpha \vdash_X \iota_X^*(\exists_{\iota_X}(\alpha))$ and $\beta \vdash_Y \iota_Y^*(\exists_{\iota_Y}(\beta))$.

The coequalizer of two morphisms $f, g : (X, \alpha) \to (Y, \beta)$ is given by $(C, \exists_q(\beta))$ where C is the coequalizer of f and g in Set – that is, the quotient of Y with respect to the smallest equivalence relation $\sim$ such that $f(x) \sim g(x)$ for all $x \in X$ – and $q : Y \to C$ is the projection onto the quotient, together with q itself which is tracked as a morphism $(Y, \beta) \to (C, \exists_q(\beta))$ since $\beta \vdash_Y q^*(\exists_q(\beta))$ by the unit of the adjunction $\exists_q \dashv q^*$. Note in particular how this implies that $\exists_q(\beta)$ is well-defined as an existence predicate: in fact, by surjectivity of q, any $c \in C$ is given by $q(y)$ for some $y \in Y$, from which $\beta(y) \to \exists_q(\beta)(q(y)) \in S$ and hence $\exists_q(\beta)(c) \in S$ as $\beta(y) \in S$.

□

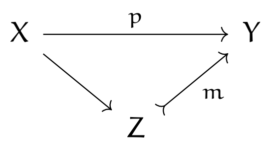
Corollary 8.7. $\Gamma : \mathsf{ArrAsm}(\mathcal{A}) \to \mathsf{Set}$ *preserves finite colimits.*

Remark 8.8. As Γ is obviously faithful, it reflects monomorphisms and epimorphisms; since it preserves finite limits and colimits, Γ also preserves monomorphisms and epimorphisms.

Therefore, a morphism of assemblies $f : (X, \alpha) \to (Y, \beta)$ is a monomorphism (resp. epimorphism) if and only if $f : X \to Y$ is injective (resp. surjective).

To show regularity, we will refer to the definition of a regular category given in [19, A1.3].

Definition 8.9. A morphism $p : X \to Y$ in a category C is a *cover* if, for any factorization:



where m is monic, m is actually an isomorphism.

Assuming C admits finite limits, the notion of cover is equivalent to that of strong epimorphism (and extremal epimorphism). In particular, covers in C are epic, and the image factorization of a morphism – by definition, the least subobject of the codomain it factors through – is given by a cover followed by a monomorphism.

In this context, C is *regular* if:

i. it is finitely complete;
ii. every morphism admits an image factorization;
iii. covers are stable under pullback,

in which case covers coincide with regular epimorphisms as well.

The following lemma characterizes covers in $\mathsf{ArrAsm}(\mathcal{A})$.

Lemma 8.10. *Let* $f : (X, \alpha) \to (Y, \beta)$ *be a morphism of assemblies. Then,* f *is a cover if and only if* f *is surjective and* $\beta \dashv\vdash_Y \exists_f(\alpha)$*.*

Proof. Suppose f is a cover. Then, f is surjective as $\mathsf{ArrAsm}(\mathcal{A})$ is finitely complete, and $\exists_f(\alpha) \vdash_Y \beta$ follows from $\alpha \vdash_Y f^*(\beta)$ by the adjunction $\exists_f \dashv f^*$. To show that $\beta \vdash_Y \exists_f(\alpha)$, consider then the factorization:

$$\begin{array}{ccc} (X,\alpha) & \xrightarrow{\quad f \quad} & (Y,\beta) \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle \mathrm{id}_Y} & \\ & (Y, \exists_f(\alpha)) & \end{array}$$

where f is tracked as a morphism $(X, \alpha) \to (Y, \exists_f(\alpha))$ since $\alpha \vdash_X f^*(\exists_f(\alpha))$ by the unit of the adjunction $\exists_f \dashv f^*$. As id_Y is a monomorphism since it is injective, it is hence an isomorphism: this means that id_Y is tracked as a morphism $(Y, \beta) \to (Y, \exists_f(\alpha))$, from which $\beta \vdash_Y \exists_f(\alpha)$.

Conversely, suppose f is surjective and $\beta \dashv\vdash_Y \exists_f(\alpha)$ and consider any factorization:

$$\begin{array}{ccc} (X,\alpha) & \xrightarrow{\quad f \quad} & (Y,\beta) \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle m} & \\ & (Z, \gamma) & \end{array}$$

where m is monic. Then, since f is surjective and m is injective, it follows that m must be bijective; let's then show that $m^{-1} : Y \to Z$ is tracked as a morphism $(Y, \beta) \to (Z, \gamma)$, hence making m is an isomorphism in $\mathsf{ArrAsm}(\mathcal{A})$. Explicitly, this amounts to:

$$\beta \vdash_Y \gamma m^{-1}$$

Since $\beta \vdash_Y \exists_f(\alpha)$, it suffices to show:

$$\exists_f(\alpha) \vdash_Y \gamma m^{-1}$$

which, by the adjunction $\exists_f \dashv f^*$, is equivalent to:

$$\alpha \vdash_X \gamma m^{-1} f = \gamma g$$

and this holds as g is a morphism $(X, \alpha) \to (Z, \gamma)$. $\square$

Note that, if $\beta \dashv\vdash_Y \beta'$, then id_Y is an isomorphism $(Y, \beta) \simeq (Y, \beta')$ in $\mathsf{ArrAsm}(\mathcal{A})$. By the previous lemma, therefore, any cover $f : (X, \alpha) \to (Y, \beta)$ can be identified up to isomorphism with $f : (X, \alpha) \to (Y, \exists_f(\alpha))$.

Corollary 8.11. *Every morphism in* $\mathsf{ArrAsm}(\mathcal{A})$ *admits an image factorization.*

Proof. Let $f : (X, \alpha) \to (Y, \beta)$ be any morphism in $\mathsf{ArrAsm}(\mathcal{A})$ and let $f = i \circ \widetilde{f}$ be the image factorization of f in Set. Then, note that $\widetilde{f}$ is tracked as a morphism $(X, \alpha) \to (f(X), \exists_f(\alpha))$ since $\alpha \vdash_X \widetilde{f}^*(\exists_f(\alpha)) = f^*(\exists_f(\alpha))$ follows by the unit of the adjunction $\exists_f \dashv f^*$. Instead, the inclusion i is tracked as a morphism $(f(X), \exists_f(\alpha)) \to (Y, \beta)$ simply because $\exists_f(\alpha) \vdash_{f(X)} \beta$ is implied by $\exists_f(\alpha) \vdash_Y \beta$ which is equivalent to $\alpha \vdash_X f^*(\beta)$. Therefore, we have the following factorization in $\mathsf{ArrAsm}(\mathcal{A})$:

$$\begin{array}{ccc} (X, \alpha) & \xrightarrow{\quad f \quad} & (Y, \beta) \\ & {\scriptstyle \widetilde{f}}\searrow \qquad \nearrow{\scriptstyle i} & \\ & (f(X), \exists_f(\alpha)) & \end{array}$$

By the previous lemma, $f : (X, \alpha) \to (f(X), \exists_f(\alpha))$ is a cover, whereas $i : (f(X), \exists_f(\alpha)) \to (Y, \beta)$ is obviously a monomorphism being injective. Therefore, $i \circ \widetilde{f}$ is the image factorization of f. $\square$

Proposition 8.12. *Covers in* $\mathsf{ArrAsm}(\mathcal{A})$ *are stable under pullback.*

Proof. Let f be a cover in $\mathsf{ArrAsm}(\mathcal{A})$, which by the previous remarks we can assume of the form $f : (X, \alpha) \to (Y, \exists_f(\alpha))$ for a surjective f. Consider the pullback of f along some morphism $g : (Z, \gamma) \to (Y, \exists_f(\alpha))$:

$$\begin{array}{ccc} (P, p_0^*(\alpha) \times p_1^*(\gamma)) & \xrightarrow{\ p_0\ } & (X, \alpha) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle f} \\ (Z, \gamma) & \xrightarrow[\ g\ ]{} & (Y, \exists_f(\alpha)) \end{array}$$

To show that p_1 is a cover, as it is obviously surjective, we only need to show that $\gamma \vdash_Z \exists_{p_1}(p_0^*(\alpha) \times p_1^*(\gamma))$. By the Frobenius condition, this is equivalent to $\gamma \vdash_Z \exists_{p_1}(p_0^*(\alpha)) \times \gamma$, i.e. to $\gamma \vdash_Z \exists_{p_1}(p_0^*(\alpha))$. By the Beck-Chevalley condition, this is then equivalent to $\gamma \vdash_Z g^*(\exists_f(\alpha))$, which is true as g is tracked as a morphism $(Z, \gamma) \to (Y, \exists_f(\alpha))$.

□

Corollary 8.13. $\mathsf{ArrAsm}(\mathcal{A})$ *is a regular category.*

Corollary 8.14. $\Gamma : \mathsf{ArrAsm}(\mathcal{A}) \to \mathsf{Set}$ *preserves regular epimorphisms; hence, it is a regular functor.*

One of the key properties in reducing the logic of a realizability topos to that of the category of assemblies over the underlying PCA is that their category is endowed with a natural number object. This is still true in the case of arrow assemblies: the following construction was given by Marcus Briët in his Master thesis [6].

Proposition 8.15. $\mathsf{ArrAsm}(\mathcal{A})$ *admits a natural number object.*

Proof. First, for $f, x \in A$, we define $f^n x$ inductively for $n \in \mathbb{N}$ by letting:

$$\begin{cases} f^0 x := x \\ f^{n+1} x := f(f^n x) \end{cases}$$

Consider now the assembly $(\mathbb{N}, \nu)$ where:

$$\nu(n) := \bigcurlywedge_{f \in A} f \to \bigcurlywedge_{x \in A} x \to \partial f^n x$$

which is well-defined as an existence predicate since, for every $n \in \mathbb{N}$:

$$(\lambda fx.f^n x)^{\mathcal{A}} = \bigcurlywedge_{f \in A} f \to \partial \bigcurlywedge_{x \in A} x \to \partial f^n x \in S$$

and hence $\nu(n) \in S$ by the properties of ∂. Then, consider $0 \in \mathbb{N}$ as a function $\{*\} \to \mathbb{N}$, which is trivially tracked as a morphism $(\{*\}, \top) \to (\mathbb{N}, \nu)$ since $\nu(0) \in S$. Finally, let $s : \mathbb{N} \to \mathbb{N}$ be the successor function, and note that:

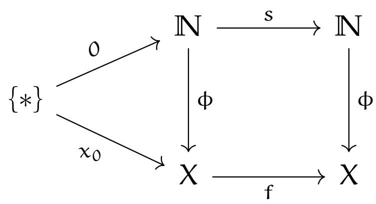
$$\begin{aligned} (\lambda nfx.\eta'(\mathbf{i}'f)(nfx))^{\mathcal{A}} &\preccurlyeq \bigcurlywedge_{n \in \mathbb{N}} \nu(n) \to \partial \bigcurlywedge_{f \in A} f \to \partial \bigcurlywedge_{x \in A} x \to \partial\eta'(\partial f)(\partial f^n x) \\ &\preccurlyeq \bigcurlywedge_{n \in \mathbb{N}} \nu(n) \to \partial \bigcurlywedge_{f \in A} f \to \partial \bigcurlywedge_{x \in A} x \to \partial\partial f^{n+1} x \in S \end{aligned}$$

from which $\curlywedge_{n \in \mathbb{N}} \nu(n) \to \nu(n+1) \in S$ by the properties of ∂, meaning that s is tracked as a morphism $(\mathbb{N}, \nu) \to (\mathbb{N}, \nu)$.

Let's show that $\langle(\mathbb{N}, \nu), 0, s\rangle$ is a natural number object in $\mathsf{ArrAsm}(\mathcal{A})$. Consider a diagram:

$$(\{*\}, \top) \xrightarrow{x_0} (X, \alpha) \xrightarrow{f} (X, \alpha)$$

in $\mathsf{ArrAsm}(\mathcal{A})$. Then, there exists a unique function $\phi : \mathbb{N} \to X$ which makes the diagram:



commute in Set; to conclude, it therefore suffices to show that ϕ is tracked as a morphism $(\mathbb{N}, \nu) \to (X, \alpha)$. To prove this, first let:

$$\bar{f} := \curlywedge_{x \in X} (\partial\alpha(x)) \to \partial\alpha(f(x))$$

and note that, being $\alpha \vdash_X \alpha f$, $\bar{f} \in S$ by the properties of ∂. Inductively, we have that $\bar{f}^n(\partial\alpha(x_0)) \preccurlyeq \partial\alpha(f^n(x_0))$ for all $n \in \mathbb{N}$:

$$\begin{aligned}
\bar{f}^0(\partial\alpha(x_0)) &\preccurlyeq \partial\alpha(f^0(x_0)) \\
\bar{f}^{n+1}(\partial\alpha(x_0)) &= \left(\curlywedge_{x \in X} (\partial\alpha(x)) \to \partial\alpha(f(x)) \right) (\bar{f}^n(\partial\alpha(x_0))) \\
&\preccurlyeq \left(\curlywedge_{x \in X} (\partial\alpha(x)) \to \partial\alpha(f(x)) \right) (\partial\alpha(f^n(x_0))) \\
&\preccurlyeq \partial\alpha(f^{n+1}(x_0))
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
(\lambda n.n\bar{f}(\partial\alpha(x_0)))^{\mathcal{A}} &\preccurlyeq \curlywedge_{n \in \mathbb{N}} \nu(n) \to \partial \left(\curlywedge_{f \in A} f \to \curlywedge_{x \in A} x \to \partial f^n x \right) \bar{f}(\partial\alpha(x_0)) \\
&\preccurlyeq \curlywedge_{n \in \mathbb{N}} \nu(n) \to \partial\partial\bar{f}^n(\partial\alpha(x_0)) \\
&\preccurlyeq \curlywedge_{n \in \mathbb{N}} \nu(n) \to \partial\partial\partial\alpha(f^n(x_0)) \\
&= \curlywedge_{n \in \mathbb{N}} \nu(n) \to \partial\partial\partial\alpha(\phi(n)) \in S
\end{aligned}$$

from which $\nu \vdash_{\mathbb{N}} \alpha\phi$ by the properties of ∂. $\square$

Remark 8.16. As a matter of fact, $\mathsf{ArrAsm}(\mathcal{A})$ is even a quasi-topos, that is, it is also locally cartesian closed and it admits a strong subobject classifier. The proof carries over to arrow algebras essentially unchanged from that in [7].

8.2 FROM ARROW ASSEMBLIES TO THE ARROW TOPOS

Let $\mathcal{A} = (A, \preccurlyeq, \to, S)$ be an arrow algebra, let $P : \mathsf{Set}^{op} \to \mathsf{HeytPre}$ be the associated arrow tripos and let $\mathsf{AT}(\mathcal{A})$ be the corresponding arrow topos.

By definition, an object of $\mathsf{AT}(\mathcal{A})$ is a set X endowed with a partial equivalence relation $\sim_X \in P(X \times X)$, so that the statement $x \sim_X x$ can be interpreted as expressing that x *exists*. Instead, an assembly over $\mathcal{A}$ consists of a set X endowed with an S-valued predicate $\alpha : X \to S$ such that $\alpha(x)$ can be thought of as a witness for the fact that x exists. Following this intuition, it seems natural to try to embed $\mathsf{ArrAsm}(\mathcal{A})$ into $\mathsf{AT}(\mathcal{A})$.

Notation. Following [34], given any $\phi \in P(X)$, we denote with $\sim_\phi$ the predicate $\exists_{\delta_X}(\phi) \in P(X \times X)$, where $\delta_X : X \to X \times X$ is the diagonal function $\langle \mathrm{id}_X, \mathrm{id}_X \rangle$. Recall then that a predicate of the form $\sim_\phi$ satisfies:

i. $P \vDash \forall x(x \sim_\phi x \leftrightarrow \phi(x))$;
ii. $P \vDash \forall x, x'(x \sim_\phi x' \to x' \sim_\phi x)$;
iii. $P \vDash \forall x, x', x''(x \sim_\phi x' \wedge x' \sim_\phi x'' \to x \sim_\phi x'')$.

For an assembly (X, α), let then $\iota(X, \alpha) := (X, \sim_\alpha)$. By the previous remark, $\sim_\alpha$ is symmetric and transitive, hence $(X, \sim_\alpha)$ is an arrow set.

Let now $f : (X, \alpha) \to (Y, \beta)$ be a morphism of assemblies. We define $\iota(f)$ as the morphism $(X, \sim_\alpha) \to (Y, \sim_\beta)$ in $\mathsf{AT}(\mathcal{A})$ represented by:

$$F := \exists_{\langle \mathrm{id}_X, f \rangle}(\alpha) \in P(X \times Y)$$

First, let us show that F is a functional relation from $(X, \sim_\alpha)$ to $(Y, \sim_\beta)$. To keep the proof reasonably short, we will make uncommented use of the property $P \vDash \forall x(x \sim_\phi x \leftrightarrow \phi(x))$ of predicates of the form $\sim_\phi$ for $\phi \in P(X)$, the Heyting adjunction in each $P(I)$, and the adjunction $\exists_f \dashv f^*$ for any function f – together with the fact that f^* is a morphism of Heyting prealgebras.

Lemma 8.17. F *is strict.*

Proof. $P \vDash \forall x, y(F(x,y) \to x \sim_X x \wedge y \sim_Y y)$ amounts to:

$$\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha) \vdash_{X\times Y} \pi_1^*(\alpha) \times \pi_2^*(\beta)$$

where π_1 and π_2 are the projections from $X \times Y$. Then:

$$\begin{aligned} \exists_{\langle \mathrm{id}_X, f\rangle}(\alpha) \vdash_{X\times Y} \pi_1^*(\alpha) \times \pi_2^*(\beta) \\ \iff \alpha \vdash_X \alpha \times f^*(\beta) \\ \iff \alpha \vdash_X f^*(\beta) \end{aligned}$$

which is true as f is a morphism of assemblies $(X, \alpha) \to (Y, \beta)$. □

Lemma 8.18. F *is relational.*

Proof. $P \vDash \forall x, x', y, y'(F(x,y) \wedge x \sim_\alpha x' \wedge y \sim_\beta y' \to F(x', y'))$ amounts to:

$$\pi_{13}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \times \pi_{12}^*(\exists_{\delta_X}(\alpha)) \times \pi_{34}^*(\exists_{\delta_Y}(\beta)) \vdash_{X\times X\times Y\times Y} \pi_{24}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha))$$

where $\pi_{12}, \pi_{13}, \pi_{24}, \pi_{34}$ are projections from $X \times X \times Y \times Y$. Then, since $\alpha \vdash_X f^*(\beta)$:

$$\begin{aligned}
&\pi_{13}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \times \pi_{12}^*(\exists_{\delta_X}(\alpha)) \times \pi_{34}^*(\exists_{\delta_Y}(\beta)) \vdash_{X\times X\times Y\times Y} \pi_{24}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \\
\iff\ & \pi_{13}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \vdash_{X\times X\times Y\times Y} \\
& \quad \pi_{12}^*(\exists_{\delta_X}(\alpha)) \to \pi_{34}^*(\exists_{\delta_Y}(\beta)) \to \pi_{24}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \\
\Longleftarrow\ & \pi_{13}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(f^*(\beta))) \vdash_{X\times X\times Y\times Y} \\
& \quad \pi_{12}^*(\exists_{\delta_X}(\alpha)) \to \pi_{34}^*(\exists_{\delta_Y}(\beta)) \to \pi_{24}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \\
\iff\ & \exists_{\langle \pi_1, \pi_2, f\pi_1, \pi_3\rangle}((f\pi_1)^*(\beta)) \vdash_{X\times X\times Y\times Y} \\
& \quad \pi_{12}^*(\exists_{\delta_X}(\alpha)) \to \pi_{34}^*(\exists_{\delta_Y}(\beta)) \to \pi_{24}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \\
\iff\ & (f\pi_1)^*(\beta) \vdash_{X\times X\times Y} \\
& \quad \langle \pi_1, \pi_2\rangle^*(\exists_{\delta_X}(\alpha)) \to \langle f\pi_1, \pi_3\rangle^*(\exists_{\delta_Y}(\beta)) \to \langle \pi_2, \pi_3\rangle^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \\
\iff\ & \langle f\pi_1, \pi_3\rangle^*\,(\exists_{\delta_Y}(\beta)) \vdash_{X\times X\times Y} \\
& \quad (f\pi_1)^*(\beta) \to \langle \pi_1, \pi_2\rangle^*\,(\exists_{\delta_X}(\alpha)) \to \langle \pi_2, \pi_3\rangle^*\,(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \\
\iff\ & \exists_{\langle \pi_1, \pi_2, f\pi_1\rangle}((f\pi_1)^*(\beta)) \vdash_{X\times X\times Y} \\
& \quad (f\pi_1)^*(\beta) \to \langle \pi_1, \pi_2\rangle^*\,(\exists_{\delta_X}(\alpha)) \to \langle \pi_2, \pi_3\rangle^*\,(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \\
\iff\ & (f\pi_1)^*(\beta) \vdash_{X\times X} (f\pi_1)^*(\beta) \to \exists_{\delta_X}(\alpha) \to \langle \pi_2, f\pi_1\rangle^*\,(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \\
\iff\ & \exists_{\delta_X}(\alpha) \vdash_{X\times X} (f\pi_1)^*(\beta) \to \langle \pi_2, f\pi_1\rangle^*\,(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \\
\iff\ & \alpha \vdash_X f^*(\beta) \to \langle \mathrm{id}_X, f\rangle^*\,(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \\
\iff\ & \alpha \vdash_X \langle \mathrm{id}_X, f\rangle^*\,(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha))
\end{aligned}$$

which is ensured by the unit of the adjunction $\exists_{\langle \mathrm{id}_X, f\rangle} \dashv \langle \mathrm{id}_X, f\rangle^*$. In particular, we have also made use of the Beck-Chevalley condition applied to the pullback squares:

$$\begin{array}{ccc} X & \xrightarrow{\langle \mathrm{id}_X, f\rangle} & X \times Y \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle f\times \mathrm{id}_Y} \\ Y & \xrightarrow[\delta_Y]{} & Y \times Y \end{array}$$

$$\begin{array}{ccc} X \times X & \xrightarrow{\langle \pi_1, \pi_2, f\pi_1\rangle} & X \times X \times Y \\ {\scriptstyle f\pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \langle f\pi_1, \pi_3\rangle} \\ Y & \xrightarrow[\delta_Y]{} & Y \times Y \end{array}$$

$$\begin{array}{ccc} X \times X \times Y & \xrightarrow{\langle \pi_1\pi_2, f\pi_1, \pi_3\rangle} & X \times X \times Y \times Y \\ & \lrcorner & \downarrow{\scriptstyle \pi_{13}} \\ {\scriptstyle f\pi_1}\downarrow & & X \times Y \\ & & \downarrow{\scriptstyle f\times \mathrm{id}_Y} \\ Y & \xrightarrow[\delta_Y]{} & Y \times Y \end{array}$$

□

Lemma 8.19. F *is single-valued.*

Proof. $P \vDash \forall x, y, y'(F(x, y) \wedge F(x, y') \to y \sim_\beta y')$ amounts to:

$$\pi_{12}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \times \pi_{13}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \vdash_{X\times Y\times Y} \pi_{23}^*(\exists_\delta(\beta))$$

where π_{12}, π_{13} and π_{23} are projections from $X \times Y \times Y$. Then, since $\alpha \vdash_X f^*(\beta)$:

$$\begin{aligned} & \pi_{12}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \times \pi_{13}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \vdash_{X\times Y\times Y} \pi_{23}^*(\exists_{\delta_Y}(\beta)) \\ \iff\ & \pi_{12}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \vdash_{X\times Y\times Y} \pi_{13}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \to \pi_{23}^*(\exists_{\delta_Y}(\beta)) \\ \Longleftarrow\ & \pi_{12}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(f^*(\beta))) \vdash_{X\times Y\times Y} \pi_{13}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \to \pi_{23}^*(\exists_{\delta_Y}(\beta)) \\ \iff\ & \pi_{12}^*(f \times \mathrm{id}_Y)^*(\exists_{\delta_Y}(\beta)) \vdash_{X\times Y\times Y} \pi_{13}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \to \pi_{23}^*(\exists_{\delta_Y}(\beta)) \\ \iff\ & \exists_{\langle \pi_1, f\pi_2, \pi_2\rangle}((f\pi_1)^*(\beta)) \vdash_{X\times Y\times Y} \pi_{13}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \to \pi_{23}^*(\exists_{\delta_Y}(\beta)) \\ \iff\ & (f\pi_1)^*(\beta) \vdash_{X\times Y} \exists_{\langle \mathrm{id}_X, f\rangle}(\alpha) \to (f \times \mathrm{id}_Y)^*(\exists_{\delta_Y}(\beta)) \\ \iff\ & \exists_{\langle \mathrm{id}_X, f\rangle}(\alpha) \vdash_{X\times Y} (f\pi_1)^*(\beta) \to (f \times \mathrm{id}_Y)^*(\exists_{\delta_Y}(\beta)) \\ \iff\ & \alpha \vdash_X f^*(\beta) \to \langle f, f\rangle^*(\exists_{\delta_Y}(\beta)) \\ \iff\ & \alpha \vdash_X \langle f, f\rangle^*(\exists_{\delta_Y}(\beta)) \\ \iff\ & \alpha \vdash_X f^*\delta_Y^*(\exists_{\delta_Y}(\beta)) \\ \Longleftarrow\ & f^*(\beta) \vdash_X f^*\delta_Y^*(\exists_{\delta_Y}(\beta)) \\ \Longleftarrow\ & \beta \vdash_Y \delta_Y^*(\exists_{\delta_Y}(\beta)) \end{aligned}$$

which is ensured by the unit of the adjunction $\exists_{\delta_Y} \dashv \delta_Y^*$. In particular, we have also made use of the Beck-Chevalley condition applied to the pullback squares:

$$\begin{array}{ccc} X & \xrightarrow{\langle \mathrm{id}_X, f\rangle} & X \times Y \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle f\times \mathrm{id}_Y} \\ Y & \xrightarrow[\delta_Y]{} & Y \times Y \end{array} \qquad \begin{array}{ccc} X \times Y & \xrightarrow{\langle \pi_1, f\pi_2, \pi_2\rangle} & X \times Y \times Y \\ & \lrcorner & \downarrow{\scriptstyle \pi_{12}} \\ {\scriptstyle f\pi_1}\downarrow & & X \times Y \\ & & \downarrow{\scriptstyle f\times \mathrm{id}_Y} \\ Y & \xrightarrow[\delta_Y]{} & Y \times Y \end{array}$$

□

Lemma 8.20. F *is total.*

Proof. $P \vDash \forall x(x \sim_\alpha x \to \exists y F(x,y))$ amounts to:

$$\alpha \vdash_X \exists_{\pi_X} \exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)$$

which holds trivially since $\exists_{\pi_X} \exists_{\langle \mathrm{id}_X, f\rangle}(\alpha) \dashv\vdash_X \exists_{\mathrm{id}_X}(\alpha) \dashv\vdash_X \alpha$. □

Let us now show that the association $(X,\alpha) \mapsto (X, \sim_\alpha)$ is functorial.

Proposition 8.21. ι *is a functor* $\mathsf{ArrAsm}(\mathcal{A}) \to \mathsf{AT}(\mathcal{A})$.

Proof. Let (X,α) be an assembly. The correspondent identity morphism in $\mathsf{ArrAsm}(\mathcal{A})$ is given by id_X, and $\iota(\mathrm{id}_X)$ is represented by $\exists_{\langle \mathrm{id}_X, \mathrm{id}_X\rangle}(\alpha) = \exists_{\delta_X}(\alpha) = \sim_\alpha$ which also represents the identity at $(X, \sim_\alpha)$ in $\mathsf{AT}(\mathcal{A})$.

Consider now $f : (X,\alpha) \to (Y,\beta)$ and $g : (Y,\beta) \to (Z,\gamma)$ in $\mathsf{ArrAsm}(\mathcal{A})$. Then, $\iota(gf)$ is represented by $\exists_{\langle \mathrm{id}_X, gf\rangle}(\alpha)$ while $\iota(g) \circ \iota(f)$ is represented by:

$$\exists_{\pi_{13}}(\pi_{12}^*(\exists_{\langle \mathrm{id}, f\rangle}(\alpha)) \times \pi_{23}^*(\exists_{\langle \mathrm{id}, g\rangle}(\beta)))$$

To show that they define the same morphism in $\mathsf{AT}(\mathcal{A})$ we need to show that they are isomorphic in $P(X \times Z)$; recall moreover that, as they are functional relations from $(X, \sim_\alpha)$ to $(Z, \sim_\gamma)$, it suffices to show only one inequality. Then, since $\alpha \vdash_X f^*(\beta)$:

$$
\begin{aligned}
&\exists_{\pi_{13}}(\pi_{12}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \times \pi_{23}^*(\exists_{\langle \mathrm{id}_Y, g\rangle}(\beta))) \vdash_{X\times Z} \exists_{\langle \mathrm{id}_X, gf\rangle}(\alpha)\\
&\iff \pi_{12}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \times \pi_{23}^*(\exists_{\langle \mathrm{id}_Y, g\rangle}(\beta)) \vdash_{X\times Y\times Z} \pi_{13}^*(\exists_{\langle \mathrm{id}_X, gf\rangle}(\alpha))\\
&\iff \pi_{12}^*(\exists_{\langle \mathrm{id}_X, f\rangle}(\alpha)) \vdash_{X\times Y\times Z} \pi_{23}^*(\exists_{\langle \mathrm{id}_Y, g\rangle}(\beta)) \to \pi_{13}^*(\exists_{\langle \mathrm{id}_X, gf\rangle}(\alpha))\\
&\iff \exists_{\langle \pi_1, f\pi_1, \pi_2\rangle}(\pi_1^*(\alpha)) \vdash_{X\times Y\times Z} \pi_{23}^*(\exists_{\langle \mathrm{id}_Y, g\rangle}(\beta)) \to \pi_{13}^*(\exists_{\langle \mathrm{id}_X, gf\rangle}(\alpha))\\
&\iff \pi_1^*(\alpha) \vdash_{X\times Z} \langle f\pi_1, \pi_2\rangle^* (\exists_{\langle \mathrm{id}_Y, g\rangle}(\beta)) \to \exists_{\langle \mathrm{id}_X, gf\rangle}(\alpha)\\
&\iff \langle f\pi_1, \pi_2\rangle^* (\exists_{\langle \mathrm{id}_Y, g\rangle}(\beta)) \vdash_{X\times Z} \pi_1^*(\alpha) \to \exists_{\langle \mathrm{id}_X, gf\rangle}(\alpha)\\
&\iff \exists_{\langle \mathrm{id}_X, gf\rangle}(f^*(\beta)) \vdash_{X\times Z} \pi_1^*(\alpha) \to \exists_{\langle \mathrm{id}_X, gf\rangle}(\alpha)\\
&\iff f^*(\beta) \vdash_X \alpha \to \langle \mathrm{id}_X, gf\rangle^* \exists_{\langle \mathrm{id}_X, gf\rangle}(\alpha)\\
&\iff \alpha \vdash_X \langle \mathrm{id}_X, gf\rangle^* \exists_{\langle \mathrm{id}_X, gf\rangle}(\alpha)
\end{aligned}
$$

which is ensured by the unit of the adjunction $\exists_{\langle \mathrm{id}_X, gf\rangle} \dashv \langle \mathrm{id}_X, gf\rangle^*$. In particular, we have also made use of the Beck-Chevalley condition applied to the pullback squares:

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\langle \pi_1, f\pi_1, \pi_2\rangle} & X \times Y \times Y\\
{\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_{12}}\\
X & \xrightarrow[\langle \mathrm{id}_X, f\rangle]{} & X \times Y
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{\langle \mathrm{id}_X, gf\rangle} & X \times Z\\
{\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle \langle f\pi_1, \pi_2\rangle}\\
Y & \xrightarrow[\langle \mathrm{id}_Y, g\rangle]{} & Y \times Z
\end{array}
$$

□

Remark 8.22. In the proofs of this section, we have not really made use of any specific property of arrow algebras, nor of the fact that existence predicates take value in the separator. Indeed, the previous discussion holds for arbitrary (canonically presented) triposes and considering arbitrary predicates $\alpha \in P(X)$ instead of existence predicates $\alpha : X \to S$.

Question. The functor ι is fully faithful in the case of arrow algebras of the form $\mathrm{Pow}(\mathbb{P})$ for a discrete and absolute PCA $\mathbb{P}$. Which conditions on an arbitrary arrow algebra suffice to recover the same result?

8.3 CONSTANT OBJECTS

As it happens for PCAs and implicative algebras, the forgetful functor $\Gamma : \mathsf{ArrAsm}(\mathcal{A}) \to \mathsf{Set}$ has a right adjoint $\nabla : \mathsf{Set} \to \mathsf{ArrAsm}(\mathcal{A})$, the *constant object* functor:

- for a set X, we let $\nabla(X) \coloneqq (X, \top_X)$, where $\top_X$ is the constant function on X of value $\top$;
- for a function $f : X \to Y$, we let $\nabla(f) \coloneqq f$, which is trivially tracked as a morphism $(X, \top_X) \to (Y, \top_Y)$.

Lemma 8.23. $\Gamma \dashv \nabla$*, and* $\Gamma\nabla = \mathrm{id}_{\mathsf{Set}}$. *In particular,* ∇ *is full and faithful.*

Proof. Indeed, $\mathsf{ArrAsm}(\mathcal{A})((X, \alpha), \nabla Y) = \mathsf{Set}(X, Y)$. □

Proposition 8.24. $\nabla : \mathsf{Set} \to \mathsf{ArrAsm}(\mathcal{A})$ *is a regular functor.*

Proof. First, let us show that ∇ preserves finite limits.

i. Of course $\nabla\{*\} = (\{*\}, \top)$ is the terminal object in $\mathsf{ArrAsm}(\mathcal{A})$.
ii. For all sets X and Y, we have:

$$\begin{aligned}\nabla(X \times Y) &= (X \times Y, \top_{X \times Y}) \\ &\simeq (X \times Y, \top_X \otimes \top_Y) \\ &\simeq (X, \top_X) \times (Y, \top_Y)\end{aligned}$$

so ∇ preserves binary products.
iii. Given the equalizer $e : E \to X$ of two functions $f, g : X \to Y$, $\nabla(e) = e : (E, \top_E) \to (X, \top_X)$ is obviously the equalizer of $\nabla(f), \nabla(g) : (X, \top_X) \to (Y, \top_Y)$.

Let now $f : X \to Y$ be a regular epimorphism in Set, that is, a surjective function. Then, $\nabla f = f : (X, \top_X) \to (Y, \top_Y)$ is obviously surjective; by the previous characterization of regular epimorphisms in $\mathsf{ArrAsm}(\mathcal{A})$ we then have to show that $\top_Y \vdash_Y \exists_f(\top_X)$. Explicitly, this means:

$$\bigcurlywedge_{y \in Y} \top \to \bigcurlywedge_{a \in A} \left(\bigcurlywedge_{x \in f^{-1}(y)} \top \to \partial a \right) \to \partial\partial a \in S$$

i.e.:

$$\top \to \bigcurlywedge_{a \in A} (\top \to \partial a) \to \partial\partial a \in S$$

which is clearly true. □

Remark 8.25. The composition $\iota\nabla : \mathsf{Set} \to \mathsf{AT}(\mathcal{A})$ is the constant object functor for the topos $\mathsf{AT}(\mathcal{A})$ as described in Chapter 2.

Question. Is $\mathsf{ArrAsm}(\mathcal{A})$ equivalent to the full subcategory of $\mathsf{AT}(\mathcal{A})$ on subobjects of constant objects, so that $\mathsf{AT}(\mathcal{A})$ is the ex/reg completion of $\mathsf{ArrAsm}(\mathcal{A})$?

8.4 FUNCTORS BETWEEN CATEGORIES OF ASSEMBLIES

As in [34] and [39], we now briefly discuss functors between categories of arrow assemblies induced by implicative morphisms. Let $f : \mathcal{A} \to \mathcal{B}$ be an implicative morphism.

We define a functor $\mathsf{ArrAsm}(f) : \mathsf{ArrAsm}(\mathcal{A}) \to \mathsf{ArrAsm}(\mathcal{B})$ as follows:

– for an assembly (X, α) over $\mathcal{A}$, we let $\mathsf{ArrAsm}(f)(X, \alpha) \coloneqq (X, f\alpha)$, which is well-defined since $f(S_A) \subseteq S_B$;

– for a morphism of assemblies $g : (X, \alpha) \to (Y, \beta)$ over $\mathcal{A}$, we let $\mathsf{ArrAsm}(f)(g) \coloneqq g$, which is tracked as a morphism $(X, f\alpha) \to (Y, f\beta)$ since $\alpha \vdash_X \beta g$ implies $f\alpha \vdash_X f\beta g$.

Lemma 8.26. $\mathsf{ArrAsm}(f)$ *is a left exact functor* $\mathsf{ArrAsm}(\mathcal{A}) \to \mathsf{ArrAsm}(\mathcal{B})$.

Proof. $\mathsf{ArrAsm}(f)$ obviously preserves the terminal object as $f(\top) \dashv\vdash \top$.

Let (X, α) and (Y, β) be assemblies over $\mathcal{A}$ and consider their product $(X \times Y, \alpha \otimes \beta)$, which $\mathsf{ArrAsm}(f)$ maps to $(X \times Y, f(\alpha \otimes \beta))$. To show that $\mathsf{ArrAsm}(f)$ preserves products, we then need to show that $f(\alpha \otimes \beta) \dashv\vdash_{X\times Y} f\alpha \otimes f\beta$; explicitly, this means:

$$\bigcurlywedge_{x\in X} \bigcurlywedge_{y\in Y} f(\alpha(x) \times \beta(y)) \to f\alpha(x) \times f\beta(y) \in S$$
$$\bigcurlywedge_{x\in X} \bigcurlywedge_{y\in Y} f\alpha(x) \times f\beta(y) \to f(\alpha(x) \times \beta(y)) \in S$$

which follow by the properties of implicative morphisms.

Let $e : (E, \alpha e) \to (X, \alpha)$ be the equalizer of two morphisms $g, g' : (X, \alpha) \to (Y, \beta)$, which $\mathsf{ArrAsm}(f)$ maps to $i : (E, f\alpha e) \to (X, f\alpha)$. Trivially, this is again the equalizer of $g, g' : (X, f\alpha) \to (Y, f\beta)$. $\square$

Remark 8.27. Clearly $\Gamma_{\mathcal{B}} \circ \mathsf{ArrAsm}(f) = \Gamma_{\mathcal{A}}$: a functor $F : \mathsf{ArrAsm}(\mathcal{A}) \to \mathsf{ArrAsm}(\mathcal{B})$ such that $\Gamma_{\mathcal{B}} \circ F \simeq \Gamma_{\mathcal{A}}$ is called a Γ-*functor*.

Similarly, $\mathsf{ArrAsm}(f) \circ \nabla_{\mathcal{A}} \simeq \nabla_{\mathcal{B}}$ since $f\top_X \dashv\vdash \top_Y$: a functor $F : \mathsf{ArrAsm}(\mathcal{A}) \to \mathsf{ArrAsm}(\mathcal{B})$ such that $F \circ \nabla_{\mathcal{A}} \simeq \nabla_{\mathcal{B}}$ is called a ∇*-functor*.

We can also extend ArrAsm to a 2-functor on ArrAlg. Given $f, g : \mathcal{A} \to \mathcal{B}$ such that $f \vdash g$, in fact, we can define a natural transformation $\xi \coloneqq \mathsf{ArrAsm}(f \vdash g) : \mathsf{ArrAsm}(f) \Rightarrow \mathsf{ArrAsm}(g)$ by letting, for each assembly (X, α) on $\mathcal{A}$:

$$\xi_{(X,\alpha)} \coloneqq \mathrm{id}_X$$

which is tracked as a morphism $(X, f\alpha) \to (X, g\alpha)$ since, if $f \vdash g$, then $f\alpha \vdash_X g\alpha$. Recall by [34, Proposition 1.6.1] that natural transformations between Γ-functors are uniquely determined if they exist, so that ξ above is the unique $\mathsf{ArrAsm}(f) \Rightarrow \mathsf{ArrAsm}(g)$.

Proposition 8.28. *If* $f : \mathcal{A} \to \mathcal{B}$ *is a regular implicative morphism, then* $\mathsf{ArrAsm}(f) : \mathsf{ArrAsm}(\mathcal{A}) \to \mathsf{ArrAsm}(\mathcal{B})$ *is a regular functor.*

Proof. As we've seen, $\mathsf{ArrAsm}(f)$ is always left exact, and therefore it is regular if and only if it preserves regular epimorphisms. Recall then that regular epimorphisms in $\mathsf{ArrAsm}(\mathcal{A})$ can be characterized as surjective morphisms $g : (X, \alpha) \to (Y, \exists_g(\alpha))$, which $\mathsf{ArrAsm}(f)$ maps to $g : (X, f\alpha) \to (Y, f\exists_g(\alpha))$, obviously surjective.

Therefore, if f is regular, then $f\exists_g(\alpha) \dashv\vdash_Y \exists_g(f\alpha)$, which means that $\mathsf{ArrAsm}(f)(g) = g : (X, f\alpha) \to (Y, f\exists_g(\alpha)) \simeq (Y, \exists_g(f\alpha))$ is a regular epimorphism in $\mathsf{ArrAsm}(\mathcal{B})$ as well. □

CONCLUSION

In this thesis, we have lifted the theory of arrow algebras to a categorical framework for the study of toposes arising from triposes in a simple and concrete way. This has been achieved by introducing various notions of morphisms between arrow algebras and studying how they correspond to transformations of the associated arrow triposes. By specializing these correspondences to the case of geometric inclusions, we have characterized subtoposes of arrow toposes completely in terms of nuclei on arrow algebras, in a generalization of the corresponding locale-theoretic notion, hence giving a positive answer to a conjecture of [2]. To further demonstrate the stability of the theory, we have studied modified realizability from the abstract point of view offered by arrow algebras, introducing a pseudofunctorial construction which greatly extends previous results known in the literature. Finally, we have defined a generalization of the traditional notion of assemblies over a PCA in the context of arrow algebras, possibly setting the ground for a systematic study of arrow toposes as ex/reg completions.

Clearly, a lot of aspects of the theory of arrow algebras and arrow toposes are still to be explored; we name here a few possible directions for research.

1. As already noted in [2], it is still unclear how exactly arrow algebras relate to other structures inducing triposes and hence toposes: particularly, basic combinatory objects of [14] and evidenced frames of [8], which also admit a notion of morphisms to compare with that of implicative morphisms.

2. In [29], Miquel showed how implicative triposes are complete with respect to Set-triposes, that is, every tripos over Set is isomorphic to an implicative tripos. Of course, this result immediately extends to arrow triposes, but it crucially requires the Axiom of Choice and therefore it may not generalize to other base toposes. It would be worth to analyze his construction from the point of view of arrow

algebras to see if there is a way to recover the same completeness result for arrow triposes in a constructive way, by defining some 'canonical' arrow algebra inducing a given tripos.

In parallel, a possible theory of arrow algebras over more general base toposes could be developed: to this aim, in this thesis, we have chosen to stick to a constructive metatheory.

3. In this thesis, we have focused on the theoretical framework for studying geometric morphisms of localic and realizability toposes from a more concrete perspective: the next step would rationally be to set this machinery in motion, possibly shedding new light on the interrelations between localic and realizability toposes.

   Coherently, it would be worth investigating if other notions of morphisms between toposes admit simple characterizations at the level of arrow algebras: for example, *localic*, *hyperconnected*, *local* and *bounded* geometric morphisms, but also *logical functors* which don't immediately fit with the theory of implicative morphisms.

   On another hand, the notion of *gluing* of toposes along a left exact functor can be recovered at the level of triposes and left exact transformations: can we also give a corresponding construction at the level of arrow algebras and implicative morphisms?

4. Finally, the biggest open question concerns the category of assemblies, which as of now does not possess most of the nice properties it has in the case of assemblies over a discrete and absolute PCA: first of all, being a full subcategory of the arrow topos, but also being (equivalent to) the subcategory of $\neg\neg$-separated objects, or having the arrow topos as its ex/reg completion.

   As already mentioned, our definition does not generalize Zoethout's definition of [39], which instead lifts all the desirable properties to the relative ordered case; this suggests that it may not be the 'correct' one. A possible redefinition, which does generalize Zoethout's, would be to let assemblies over an arrow algebra be pairs (X, α) where α is a function $X \to A \setminus \{\bot\}$, while morphisms are still required to be tracked by an element of the separator; however, such a definition would clearly not be ideal in a constructive metatheory. We leave the question open for future research.

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