Toposes with enough points as categories of étale spaces

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Topology and convergence of ultrafilters

Theorem (Barr)

The ultrafilter monad $\beta\colon \mathbf{Set} \to \mathbf{Set}$ *extends to a skew monad* $\underline{\beta}$ *on* **Rel** *such that* $\mathbf{Top} \cong \mathbf{LaxAlg}_{r,co}(\underline{\beta})$.

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Concretely, this means that a topology on a set X can be equivalently specified by a relation $\xi\colon \beta X \nrightarrow X$ satisfying

$$\begin{tikzcd}
X \arrow[rr, equal] \arrow[dr, "\eta_X"'] & \rotatebox[origin=c]{90}{$\supseteq$} & X \\
& \beta X \arrow[ur, "\xi"', "|" marking] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta^2 X \arrow[r, "\underline{\beta}\xi", "|" marking] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\rotatebox[origin=c]{45}{$\supseteq$}"] & \beta X \arrow[d, "\xi", "|" marking] \\
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Concretely, this means that a topology on a set X can be equivalently specified by a relation $\xi \colon X \times \beta X \to \mathbf{2}$ satisfying

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which we can think of as a notion of convergence. In particular:

- $\xi(x, \nu) \iff \forall U \subseteq X$ open, if $x \in U$ then $U \in \nu$;
- U open $\iff \forall x \in U, \nu \in \beta X,$ if $\xi(x, \nu)$ then $U \in \nu$.

Topology and convergence of ultrafilters

Remark

We recover Manes' theorem, **CompHaus** $\cong$ **Alg**(β), restricting to those topological spaces X whose convergence relation ξ is functional. Indeed:

- X is compact $\iff$ for every $\nu \in \beta X$ there exists *some* $x \in X$ such that $\xi(x, \nu)$;
- X is Hausdorff $\iff$ for every $\nu \in \beta X$ there exists *at most one* $x \in X$ such that $\xi(x, \nu)$.

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Notation

For $f \colon I \to X$ and $\nu \in \beta I$, we write $x \rightsquigarrow \lim_{i \to \nu} f(i)$ in case $\xi(x, \beta f(\nu))$ holds.

Now: one dimension higher!

Toposes with enough points

Categorifying, the role of topological spaces is played by (Grothendieck) toposes satisfying a condition akin to *spatiality* for locales.

Definition

A topos $\mathcal{E}$ has enough points if it admits a *separating set of points*, i.e. a set $X \subseteq \mathrm{pt}(\mathcal{E})$ such that the induced functor $\mathcal{E} \longrightarrow \mathbf{Set}^X$ is conservative.

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Question

Can we recover such a topos $\mathcal{E}$, up to equivalence, from its category of points $\mathrm{pt}(\mathcal{E})$ together with appropriate structure inspired by Barr's convergence relations?

Ultracategories

A partial answer to this question comes from Makkai's ultracategories.

Theorem (Makkai, Lurie)

For a coherent topos $\mathcal{E}$, the category $\mathrm{pt}(\mathcal{E})$ admits a natural structure of an ultracategory such that $\mathcal{E} \simeq \mathbf{UltCat}(\mathrm{pt}(\mathcal{E}), \mathbf{Set})$.

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Intuitively, an ultracategory is a category C endowed with abstract ultraproducts, i.e. a functorial choice of an object $\prod_{i:\nu} c_i$ in C for each set I, each I-indexed family of objects $(c_i)_{i \in I}$ in C, and each ultrafilter $\nu \in \beta I$, behaving as ultraproducts of sets.

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Ultracategories categorify compact Hausdorff spaces

$\mathbf{CompHaus} \hookrightarrow \mathbf{UltCat}$ as the small and discrete ultracategories.

Ultracategories

More formally, we can define ultracategories as algebras for Rosolini's ultracompletion pseudomonad $\mathbb{\beta}\colon \mathbf{CAT} \longrightarrow \mathbf{CAT}$, thus hard-wiring a version of Manes' theorem:

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For a category C, the category $\mathbb{β}C$ has:

- as objects, triples (I, y, ν) of a set I, a function $y\colon I \to C$, and an ultrafilter $\nu \in \beta I$, called *ultrafamilies*;
- as morphisms $(I, y, \nu) \to (I', y', \nu')$, pairs of a function $h\colon I' \to I$ such that $\beta h(\nu') = \nu$ and a family of arrows $(\alpha_i\colon y_{h(i)} \to y'_i)_{i \in I'}$ in C, both considered up to ν'-equivalence.

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Ultracategories of points

For a coherent topos $\mathcal{E}$, the category $\mathrm{pt}(\mathcal{E})$ is an ultracategory: the ultraproduct $\prod_{i:\nu} p_i$ of an ultrafamily (I, p, ν) of points is the functor

$$\mathcal{E} \xrightarrow{\langle p_i \rangle_{i \in I}} \mathbf{Set}^I \xrightarrow{\prod_{i:\nu}(-)} \mathbf{Set}$$

which is a point of $\mathcal{E}$ by Łoś's theorem.

Ultraconvergence spaces

For an arbitrary topos, Łoś's theorem fails. Inspired by Barr's extension of Manes' theorem, we thus generalize ultracategories by replacing a $\mathbb{\beta}$-algebra functor with a *profunctor*, which simultaneously allows us to categorify the description of topological spaces in terms of convergence.

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- for every $x \in X$, an *identity* ultra-arrow $\mathrm{id}_x \colon x \rightsquigarrow \lim_{* \to 1} x$;
- for every ultra-arrow $r \colon x \rightsquigarrow \lim_{i \to \mu} y_i$ and every ultrafamily of ultra-arrows $(s_i \colon y_i \rightsquigarrow \lim_{j \to \nu_i} z_{i,j})_{i \to \mu}$, a *composite* ultra-arrow $(s_i)_{i \to \mu} \cdot r \colon x \rightsquigarrow \lim_{(i,j) \to \sum_{i \to \mu} \nu_i} z_{i,j}$,

satisfying unitality and associativity axioms.

Continuous maps

Similarly, we can categorify the notion of continuity to this **Set**-valued convergence relation, which now becomes *structure* rather than *property*.

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Definition

A continuous map of ultraconvergence spaces is a functor $f\colon X \longrightarrow X'$ together with a family of functions

$$\begin{aligned} \Xi(x,(I,y,\nu)) &\longrightarrow \Xi'(f(x),(I,fy,\nu)) \\ r\colon x \rightsquigarrow \lim_{i\to\nu} y_i &\longmapsto f(r)\colon f(x) \rightsquigarrow \lim_{i\to\nu} f(y_i) \end{aligned}$$

satisfying some equational axioms.

With appropriate 2-cells, ultraconvergence spaces define a 2-category **UltSp**.

Examples

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$$\begin{tikzcd}
\mathbf{UltCat} \arrow[r, hook] & \mathbf{UltSp} \\
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The main theorem

The notion of ultraconvergence space allows us to obtain a reconstruction theorem for toposes with enough points.

Theorem (Saadia; Hamad; van Gool, Marquès, T.)

If $\mathcal{E}$ *is a topos with enough points, then* $\mathcal{E} \simeq \mathbf{UltSp}(\mathrm{pt}(\mathcal{E}), \mathbf{Set})$*.*

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In other words, restricting to ultracategories and ultraconvergence spaces such that homming into **Set** yields a topos:

$$\begin{tikzcd}
\mathbf{Topos}_{wep} \arrow[r, bend right, "\mathrm{pt}(-)"'] \arrow[r, phantom, "\perp"] & \mathbf{UltSp}_{*} \arrow[l, bend right, "\mathbf{UltSp}(-,\mathbf{Set})"'] \\
\mathbf{Topos}_{coh} \arrow[u, hook] \arrow[r, bend right, "\mathrm{pt}(-)"'] \arrow[r, phantom, "\perp"] & \mathbf{UltCat}_{*} \arrow[u, hook] \arrow[l, bend right, "\mathbf{UltCat}(-,\mathbf{Set})"']
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Étale maps over B form a category $\mathrm{Et}(B)$, equivalent to $\mathbf{UltSp}(B, \mathbf{Set})$.

The ultraconvergence space of points of a topos

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$$\mathcal{E} \xrightarrow{\langle y_i \rangle_{i \in I}} \mathbf{Set}^I \xrightarrow{\prod_{i:\nu}(-)} \mathbf{Set}$$

- For every object $\varphi \in \mathcal{E}$, we can define an étale space $\pi_\varphi \colon [\![\varphi]\!] \longrightarrow X$ where:
  - the fiber of π_φ at $x \in X$ is given by $x(\varphi)$;
  - an ultra-arrow $(x, v) \rightsquigarrow \lim_{i \to \nu}(y_i, w_i)$ in $[\![\varphi]\!]$ is given by an ultra-arrow $r \colon x \rightsquigarrow \lim_{i \to \nu} y_i$ in X such that $r_\varphi(v) = (w_i)_{i \to \nu}$.

  This assignment defines the *evaluation functor* $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$.

Reconstruction theorem

Theorem

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Although we need X to be small to prove the above result, it follows easily that $\mathcal{E} \simeq \mathrm{Et}(\mathrm{pt}(\mathcal{E}))$. In logical terms, this reads as the following reconstruction result.

Theorem

Let $\mathbb{T}$ *be a geometric theory which is complete with respect to its* **Set**-*models. Then,* $\mathrm{Mod}(\mathbb{T})$ *is an ultraconvergence space by setting ultra-arrows* $M \rightsquigarrow \lim_{i \to \nu} N_i$ *to be structure morphisms* $M \to \prod_{i:\nu} N_i$*, and* $\mathrm{Et}(\mathrm{Mod}(\mathbb{T}))$ *is the classifying topos of* $\mathbb{T}$*.*

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The localic/propositional case

In particular, if a localic topos $\mathcal{E}$ has enough points, i.e. $\mathcal{E} \simeq \mathrm{Sh}(\mathcal{O}(X))$ for some topological space X, then $\mathcal{E} \simeq \mathrm{Et}(X)$.

Proof sketch

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1. $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is full on subobjects: every subobject of $\pi_\varphi \colon [\![\varphi]\!] \longrightarrow X$ in $\mathrm{Et}(X)$ is the restriction of π_φ to $[\![\psi]\!] \subseteq [\![\varphi]\!]$ for some subobject $\psi \rightarrowtail \varphi$ in $\mathcal{E}$.

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2. $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is covering: every étale space $p \colon Y \longrightarrow X$ is covered by an epimorphism $\alpha \colon \pi_\varphi \twoheadrightarrow p$ in $\mathrm{Et}(X)$ for some object $\varphi \in \mathcal{E}$.

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Two points of view

Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $[\![-]\!]$. However, we can also interpret (1) as stating that $[\![-]\!]$ defines a hyperconnected geometric morphism, and (2) as stating that it defines a localic geometric morphism.

Ultraconvergence spaces as algebras

The inspiration from Barr's theorem can be pushed even further.

Theorem (Aristote, T.)

The ultracompletion pseudomonad $\mathbb{\beta}$ *extends to a skew monad* $\underline{\mathbb{\beta}}\colon \mathbf{PROF} \longrightarrow \mathbf{PROF}$, *and* $\mathbf{UltSp} \cong \mathbf{LaxAlg}^{norm}_{r,co}(\underline{\mathbb{\beta}})$.

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This means that an ultraconvergence structure on a discrete category X can be equivalently specified by a profunctor $\Xi\colon \beta X \nrightarrow X$ and two transformations

$$\begin{tikzcd}
X \arrow[rr, equal] \arrow[dr, "\eta_X"'] & \arrow[d, phantom, "\Downarrow"] & X \\
& \beta X \arrow[ur, "|" marking, "\Xi"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta^2 X \arrow[r, "\underline{\beta}\Xi", "|" marking] \arrow[d, "\mu_X"'] & \beta X \arrow[d, "\Xi", "|" marking] \arrow[dl, phantom, "\Swarrow"] \\
\beta X \arrow[r, "\Xi"', "|" marking] & X
\end{tikzcd}$$

satisfying usual coherence axioms.

$$\begin{tikzcd}
& \mathbf{UltSp} \arrow[rr, leftrightarrow, "\cong" description] & & \mathbf{LaxAlg}^{norm}_{r,co}(\underline{\mathbb{\beta}}) \\
\mathbf{UltCat} \arrow[ur, hook] \arrow[rr, leftrightarrow, "\triangleq" description] & & \mathbf{PsAlg}_{co}(\mathbb{\beta}) \arrow[ur, hook] & \\
& \mathbf{Top} \arrow[uu, hook] \arrow[rr, leftrightarrow, "\cong" description] & & \mathbf{LaxAlg}_{r,co}(\underline{\beta}) \arrow[uu, hook] \\
\mathbf{CompHaus} \arrow[uu, hook] \arrow[ur, hook] \arrow[rr, leftrightarrow, "\cong" description] & & \mathbf{Alg}(\beta) \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Future work

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- Towards step (2) of our proof, we prove a kind of Beth definability theorem for geometric logic. What does this perspective entail?
- Can we describe the equivalences induced by the two adjuctions?

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Thank you!