Toposes with enough points as categories of étale spaces

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7th Workshop on Formal Topology, 13th April 2026

Topology and convergence of ultrafilters

Theorem (Barr)

The ultrafilter monad $\beta\colon$ **Set** $\to$ **Set** *extends to a skew monad* $\underline{\beta}$ *on* **Rel** *such that* **Top** $\cong$ **LaxAlg**$_{r,co}(\underline{\beta})$.

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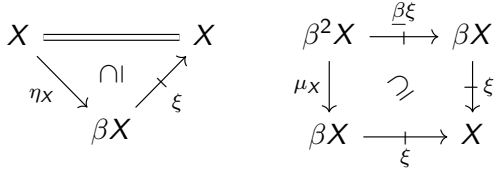

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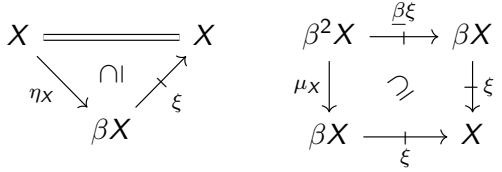

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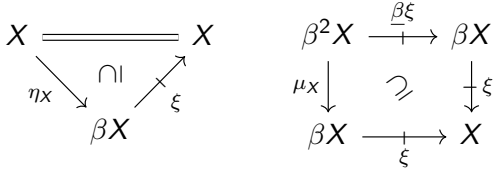

which we can think of as a notion of convergence. In particular:

- $\xi(x,\nu) \iff \forall U \subseteq X$ open, if $x \in U$ then $U \in \nu$;
- U open $\iff \forall x \in U, \nu \in \beta X$, if $\xi(x,\nu)$ then $U \in \nu$.

Topology and convergence of ultrafilters

Remark

We recover Manes' theorem, **CompHaus** $\cong$ **Alg**(β), restricting to those topological spaces X whose convergence relation ξ is functional. Indeed:

- X is compact $\iff$ for every $\nu \in \beta X$ there exists *some* $x \in X$ such that $\xi(x, \nu)$;
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Notation

For $f : I \to X$ and $\nu \in \beta I$, we write $x \rightsquigarrow \lim_{i \to \nu} f(i)$ in case $\xi(x, \beta f(\nu))$ holds.

Now: one dimension higher!

Toposes with enough points

Categorifying, the role of topological spaces is played by (Grothendieck) toposes satisfying a condition akin to *spatiality* for locales.

Definition

A topos $\mathcal{E}$ has enough points if it admits a *separating set of points*, i.e. a set $X \subseteq \mathrm{pt}(\mathcal{E})$ such that the induced functor $\mathcal{E} \longrightarrow \mathbf{Set}^X$ is conservative.

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Question

Can we recover such a topos $\mathcal{E}$, up to equivalence, from its category of points $\mathrm{pt}(\mathcal{E})$ together with appropriate structure inspired by Barr's convergence relations?

Ultracategories

A partial answer to this question comes from Makkai's ultracategories.

Theorem (Makkai, Lurie)

For a coherent topos $\mathcal{E}$, the category $\mathrm{pt}(\mathcal{E})$ *admits a natural structure of an ultracategory such that* $\mathcal{E} \simeq \mathbf{UltCat}(\mathrm{pt}(\mathcal{E}), \mathbf{Set})$.

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Intuitively, an ultracategory is a category C endowed with abstract ultraproducts, i.e. a functorial choice of an object $\prod_{i:\nu} c_i$ in C for each set I, each I-indexed family of objects $(c_i)_{i \in I}$ in C, and each ultrafilter $\nu \in \beta I$, behaving as ultraproducts of sets.

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Ultracategories categorify compact Hausdorff spaces

$\mathbf{CompHaus} \hookrightarrow \mathbf{UltCat}$ as the small and discrete ultracategories.

Ultracategories

More formally, we can define ultracategories as algebras for Rosolini's ultracompletion pseudomonad $\beth\colon \mathbf{CAT} \longrightarrow \mathbf{CAT}$, thus hard-wiring a version of Manes' theorem:

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For a category C, the category $\beth C$ has:

- as objects, triples (I, y, ν) of a set I, a function $y\colon I \to C$, and an ultrafilter $\nu \in \beta I$, called *ultrafamilies*;
- as morphisms $(I, y, \nu) \to (I', y', \nu')$, pairs of a function $h\colon I' \to I$ such that $\beta h(\nu') = \nu$ and a family of arrows $(\alpha_i\colon y_{h(i)} \to y'_i)_{i \in I'}$ in C, both considered up to ν'-equivalence.

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Ultracategories of points

For a coherent topos $\mathcal{E}$, the category $\mathrm{pt}(\mathcal{E})$ is an ultracategory: the ultraproduct $\prod_{i:\nu} p_i$ of an ultrafamily (I, p, ν) of points is the functor

$$\mathcal{E} \xrightarrow{\langle p_i \rangle_{i \in I}} \mathbf{Set}^I \xrightarrow{\prod_{i:\nu}(-)} \mathbf{Set}$$

which is a point of $\mathcal{E}$ by Łoś's theorem.

Ultraconvergence spaces

For an arbitrary topos, Łoś's theorem fails. Inspired by Barr's extension of Manes' theorem, we thus generalize ultracategories by replacing a β-algebra functor with a *profunctor*, which simultaneously allows us to categorify the description of topological spaces in terms of convergence.

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- for every $x \in X$, an *identity* ultra-arrow $\mathrm{id}_x\colon x \rightsquigarrow \lim_{*\to 1} x$;
- for every ultra-arrow $r\colon x \rightsquigarrow \lim_{i\to\mu} y_i$ and every ultrafamily of ultra-arrows $(s_i\colon y_i \rightsquigarrow \lim_{j\to\nu_i} z_{i,j})_{i\to\mu}$, a *composite* ultra-arrow $(s_i)_{i\to\mu}\cdot r\colon x \rightsquigarrow \lim_{(i,j)\to\sum_{i\to\mu}\nu_i} z_{i,j}$,

satisfying unitality and associativity axioms.

Continuous maps

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Definition

A continuous map of ultraconvergence spaces is a functor $f\colon X \longrightarrow X'$ together with a family of functions

$$\Xi(x,(I,y,\nu)) \longrightarrow \Xi'(f(x),(I,fy,\nu))$$
$$r\colon x \rightsquigarrow \lim_{i\to\nu} y_i \longmapsto f(r)\colon f(x) \rightsquigarrow \lim_{i\to\nu} f(y_i)$$

satisfying some equational axioms.

With appropriate 2-cells, ultraconvergence spaces define a 2-category **UltSp**.

Examples

- Every topological space is a $\mathbf{2}$-valued ultraconvergence space with the usual convergence relation.

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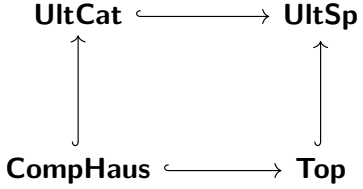

The main theorem

The notion of ultraconvergence space allows us to obtain a reconstruction theorem for toposes with enough points.

Theorem (Saadia; Hamad; van Gool, Marquès, T.)

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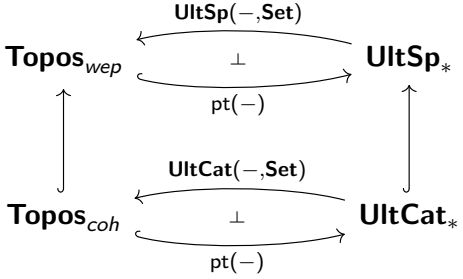
In other words, restricting to ultracategories and ultraconvergence spaces such that homming into **Set** yields a topos:



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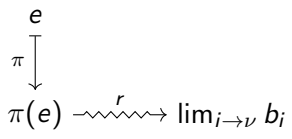

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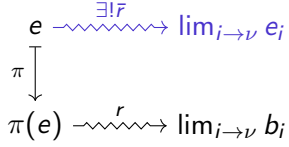

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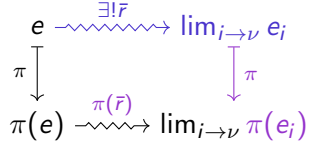

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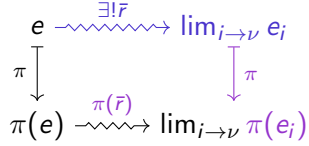

Étale maps over B form a category $\mathrm{Et}(B)$, equivalent to $\mathbf{UltSp}(B, \mathbf{Set})$.

The ultraconvergence space of points of a topos

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- For every object $\varphi \in \mathcal{E}$, we can define an étale space $\pi_\varphi \colon [\![\varphi]\!] \longrightarrow X$ where:
  - the fiber of π_φ at $x \in X$ is given by $x(\varphi)$;
  - an ultra-arrow $(x, v) \rightsquigarrow \lim_{i \to \nu}(y_i, w_i)$ in $[\![\varphi]\!]$ is given by an ultra-arrow $r \colon x \rightsquigarrow \lim_{i \to \nu} y_i$ in X such that $r_\varphi(v) = (w_i)_{i \to \nu}$.

  This assignment defines the *evaluation functor* $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$.

Reconstruction theorem

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Theorem

Let $\mathbb{T}$ be a geometric theory which is complete with respect to its **Set***-models. Then,* $\mathrm{Mod}(\mathbb{T})$ *is an ultraconvergence space by setting ultra-arrows $M \rightsquigarrow \lim_{i \to \nu} N_i$ to be structure morphisms $M \to \prod_{i:\nu} N_i$, and* $\mathrm{Et}(\mathrm{Mod}(\mathbb{T}))$ *is the classifying topos of $\mathbb{T}$.*

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The localic/propositional case

In particular, if a localic topos $\mathcal{E}$ has enough points, i.e. $\mathcal{E} \simeq \mathrm{Sh}(\mathcal{O}(X))$ for some topological space X, then $\mathcal{E} \simeq \mathrm{Et}(X)$.

Proof sketch

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1. $[\![-]\!]\colon \mathcal{E} \longrightarrow \mathsf{Et}(X)$ is full on subobjects: every subobject of $\pi_\varphi\colon [\![\varphi]\!] \longrightarrow X$ in $\mathsf{Et}(X)$ is the restriction of π_φ to $[\![\psi]\!] \subseteq [\![\varphi]\!]$ for some subobject $\psi \rightarrowtail \varphi$ in $\mathcal{E}$.

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2. $[\![-]\!]\colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is covering: every étale space $p\colon Y \longrightarrow X$ is covered by an epimorphism $\alpha\colon \pi_\varphi \twoheadrightarrow p$ in $\mathrm{Et}(X)$ for some object $\varphi \in \mathcal{E}$.

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Our proof is substantially different from both Saadia's and Hamad's, who use Butz-Moerdijk's representation theorem for toposes with enough points. Instead, we proceed similarly to Makkai's original work, in two main steps.

1. $[\![-]\!]\colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is full on subobjects: every subobject of $\pi_\varphi\colon [\![\varphi]\!] \longrightarrow X$ in $\mathrm{Et}(X)$ is the restriction of π_φ to $[\![\psi]\!] \subseteq [\![\varphi]\!]$ for some subobject $\psi \rightarrowtail \varphi$ in $\mathcal{E}$.
2. $[\![-]\!]\colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is covering: every étale space $p\colon Y \longrightarrow X$ is covered by an epimorphism $\alpha\colon \pi_\varphi \twoheadrightarrow p$ in $\mathrm{Et}(X)$ for some object $\varphi \in \mathcal{E}$.

Two points of view

Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $[\![-]\!]$.

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Two points of view

Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $[\![-]\!]$. However, we can also interpret (1) as stating that $[\![-]\!]$ defines a hyperconnected geometric morphism, and (2) as stating that it defines a localic geometric morphism.

Ultraconvergence spaces as algebras

The inspiration from Barr's theorem can be pushed even further.

Theorem (Aristote, T.)

The ultracompletion pseudomonad $\beth\colon \mathbf{CAT} \longrightarrow \mathbf{CAT}$ *extends to a skew monad* $\underline{\beth}$ *on* $\mathbf{PROF}$ *such that* $\mathbf{UltSp} \cong \mathbf{LaxAlg}^{norm}_{r,co}(\underline{\beth})$.

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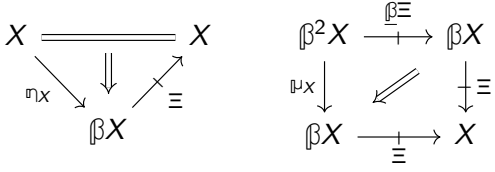
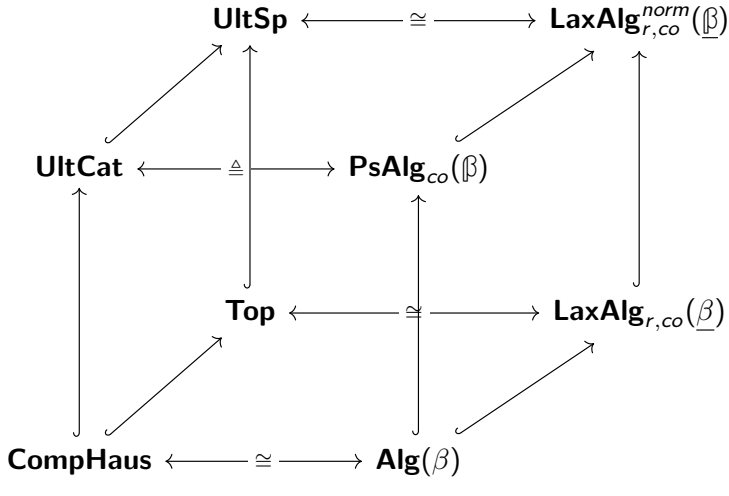
This means that an ultraconvergence structure on a discrete category X can be equivalently specified by a profunctor $\Xi\colon \beth X \nrightarrow X$ and two transformations



satisfying usual coherence axioms.


UltSp
≅
LaxAlg$^{norm}_{r,co}(\underline{\beth})$
UltCat
≜
PsAlg$_{co}(\beth)$
Top
≅
LaxAlg$_{r,co}(\underline{\beta})$
CompHaus
≅
Alg(β)

Future work

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- Towards step (2) of our proof, we prove a kind of Beth definability theorem for geometric logic. What does this perspective entail?
- Can we describe the equivalences induced by the two adjuctions?

- M. Barr (1970). "Relational algebras". In: *Reports of the Midwest Category Seminar IV*. Ed. by S. MacLane et al. Berlin, Heidelberg: Springer, pp. 39–55.
- M. Makkai (1987). "Stone duality for first order logic". In: *Advances in Mathematics* 65.2, pp. 97–170.
- J. Lurie (2018). *Ultracategories*. Available online.
- R. Garner (2020). "Ultrafilters, finite coproducts and locally connected classifying toposes". In: *Annals of Pure and Applied Logic* 171.10, p. 102831.
- G. Rosolini (2024). *Ultracompletions*. Talk at CT2024.
- G. Saadia (2025). *Extending conceptual completeness via virtual ultracategories*. arXiv: 2506.23935 [math.CT].
- A. Hamad (2025). *Generalised ultracategories and conceptual completeness of geometric logic*. arXiv: 2507.07922 [math.CT].
- S. van Gool, J. Marquès, and U. Tarantino (2025). *Toposes with enough points as categories of étale spaces*. arXiv: 2508.09604 [math.CT].
- Quentin Aristote and Umberto Tarantino (2026). *Profunctorial algebras*. arXiv: 2601.22721 [math.CT].

Thank you!