Toposes with enough points as categories of étale spaces

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Compact Hausdorff spaces and convergence

Theorem (Manes)

CompHaus $\cong \mathrm{Alg}(\beta)$, *where* $\beta\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ *is the ultrafilter monad.*

This means that, for a compact Hausdorff space X, every function $f\colon I \longrightarrow X$ extends to a function $f^*\colon \beta I \longrightarrow X$ which we can think of as computing the *limit* of f with respect to each $\nu \in \beta I$. Concretely:

$$\begin{tikzcd} \beta I \arrow[dr, "f^*"] & \\ I \arrow[u, "\eta_I"] \arrow[r, "f"'] & X \end{tikzcd} \qquad f^*(\nu) = x \iff \forall U \subseteq X \text{ open, if } x \in U \text{ then } f^{-1}(U) \in \nu$$

In particular, the algebra map $\mathrm{id}_X^*\colon \beta X \longrightarrow X$ specifies, for each ultrafilter ν on X, the unique point of X all of whose open neighborhoods lie in ν.

Topological spaces and generalized convergence

For an arbitrary topological space X, these limits may not exist nor be unique, so that the previous definition of id_X^* determines a relation $\beta X \nrightarrow X$.

Theorem (Barr)

The ultrafilter monad β extends to a monad $\underline{\beta}$: **Rel** $\longrightarrow$ **Rel**, *and* **Top** $\cong \mathrm{LaxAlg}(\underline{\beta})$.

This means that a topology on a set X can be equivalently specified by a relation $\xi\colon \beta X \nrightarrow X$ such that:

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\begin{tikzcd}
X \arrow[rr, equal] \arrow[dr, "\eta_X"'] & \arrow[d, phantom, "\subseteq"] & X \\
& \beta X \arrow[ur, "\xi"', "|" marking] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta^2 X \arrow[r, "\underline{\beta}\xi", "|" marking] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\supseteq"] & \beta X \arrow[d, "\xi", "|" marking] \\
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\begin{tikzcd}
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\beta X \arrow[r, "|" marking, "\xi"'] & X
\end{tikzcd}
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Notation

For $f\colon I \to X$ and $\nu \in \beta I$, we write $x \rightsquigarrow \lim_{i \to \nu} f(i)$ in case $\xi(x, \beta f(\nu))$ holds.

Now: one dimension higher!

Ultracategories and convergence of ultrafamilies

Going one dimension higher, the role of β is played by the *ultracompletion pseudomonad* $\mathbb{\beta}\colon \mathbf{CAT} \longrightarrow \mathbf{CAT}$. For a category C, the category $\mathbb{\beta}C$ has:

- as objects, triples (I, y, ν) of a set I, a functor $y\colon I \to C$, and an ultrafilter $\nu \in \beta I$;
- as morphisms $(I, y, \nu) \to (I', y', \nu')$, pairs of a function $h\colon I' \to I$ such that $\beta h(\nu') = \nu$ and a family of arrows $(\alpha_i\colon y_{h(i)} \to y'_i)_{i \in I'}$ in C, both considered up to ν'-equivalence.

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Intuitively, an ultracategory is a category C endowed with a functor $\Phi\colon \upbeta C \longrightarrow C$, assigning a unique *limit* in C to each *ultrafamily* (I, y, ν) in C. Formally, we define:

$$\mathbf{UltCat} := \mathrm{PsAlg}(\upbeta)$$

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Ultracategories categorify compact Hausdorff spaces

$\mathbf{CompHaus} \hookrightarrow \mathbf{UltCat}$ as those algebras whose carrier category is small and discrete.

Ultracategories and coherent toposes

Ultracategories were originally introduced by Makkai to prove a reconstruction theorem for (coherent) first-order logic.

Theorem (Makkai; Lurie)

Let $\mathbb{T}$ be a coherent theory. Then, $\mathrm{Mod}(\mathbb{T})$ is an ultracategory by setting the limit of an ultrafamily (I, M_-, ν) of models to be their ultraproduct $\prod_{i \to \nu} M_i$, and $\mathbf{UltCat}(\mathrm{Mod}(\mathbb{T}), \mathbf{Set})$ is the classifying topos of $\mathbb{T}$.

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Identifying coherent theories with coherent toposes, and restricting to the subcategory $\mathbf{UltCat}_*$ of ultracategories C such that $\mathbf{UltCat}(C, \mathbf{Set})$ is a topos, we have:

$$\begin{tikzcd} \mathbf{Topos}_{coh} \arrow[r, hook, bend right, "\mathrm{pt}(-)"'] \arrow[r, phantom, "\perp"] & \mathbf{UltCat}_* \arrow[l, bend right, "\mathbf{UltCat}(-{,}\mathbf{Set})"'] \end{tikzcd}$$

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$$\mathbf{Set}^{I} \xrightarrow{\prod_{i \to \nu}(-)} \mathbf{Set}$$

are coherent, but not necessarily geometric functors.

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\mathbf{Topos}_{wep} \arrow[rr, hook, bend right, "\mathrm{pt}(-)"'] \arrow[rr, phantom, "\perp"] & & {\color{magenta}\mathbf{?}} \arrow[ll, dotted, bend right] \\
\\
\mathbf{Topos}_{coh} \arrow[uu, hook] \arrow[rr, hook, bend right, "\mathrm{pt}(-)"'] \arrow[rr, phantom, "\perp"] & & \mathbf{UltCat} \arrow[ll, dotted, bend right] \arrow[uu, hook]
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$$\begin{tikzcd} \mathbf{Topos}_{wep} \arrow[r, hook, bend right, "\mathrm{pt}(-)"'] \arrow[r, phantom, "\perp"] & \mathbf{UltSp} \arrow[l, dotted, bend right] \\ \mathbf{Topos}_{coh} \arrow[u, hook] \arrow[r, hook, bend right, "\mathrm{pt}(-)"'] \arrow[r, phantom, "\perp"] & \mathbf{UltCat} \arrow[u, hook] \arrow[l, dotted, bend right] \end{tikzcd}$$

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- for every $x \in X$, an *identity* ultra-arrow $\mathrm{id}_x \colon x \rightsquigarrow \lim_{* \to 1} x$;
- for every ultra-arrow $r \colon x \rightsquigarrow \lim_{i \to \mu} y_i$ and every ultrafamily of ultra-arrows $(s_i \colon y_i \rightsquigarrow \lim_{j \to \nu_i} z_{i,j})_{i \to \mu}$, a *composite* ultra-arrow $(s_i)_{i \to \mu} \cdot r \colon x \rightsquigarrow \lim_{(i,j) \to \sum_{i \to \mu} \nu_i} z_{i,j}$,

satisfying some equational axioms.

Continuous maps

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Definition

A continuous map of ultraconvergence spaces is a functor $f\colon X \longrightarrow X'$ together with a family of functions

$$\begin{aligned} \Xi(x,(I,y,\nu)) &\longrightarrow \Xi'(f(x),(I,fy,\nu)) \\ r\colon x \rightsquigarrow \lim_{i\to\nu} y_i &\longmapsto f(r)\colon f(x) \rightsquigarrow \lim_{i\to\nu} f(y_i) \end{aligned}$$

also satisfying some equational axioms.

With appropriate 2-cells, ultraconvergence spaces define a 2-category **UltSp**.

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$$\begin{tikzcd}
\mathbf{UltCat} \arrow[r, hook] & \mathbf{UltSp} \\
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The main theorem

As promised, the notion of ultraconvergence space allows us to obtain a reconstruction theorem for geometric logic: in topos-theoretical terms, it reads as follows.

Theorem (Saadia; Hamad; van Gool, Marquès, T.)

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In other words, restricting to the subcategory $\mathbf{UltSp}_*$ of ultraconvergence spaces X such that $\mathbf{UltSp}(X, \mathbf{Set})$ is a topos, we have what we wanted:

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Étale maps over B form a category $\mathrm{Et}(B)$, equivalent to $\mathbf{UltSp}(B, \mathbf{Set})$.

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- For every object $\varphi \in \mathcal{E}$, we can define an étale space $\pi_\varphi \colon [\![\varphi]\!] \longrightarrow X$ where:
  - the fiber of π_φ at $x \in X$ is given by $x(\varphi)$;
  - an ultra-arrow $(x, v) \rightsquigarrow \lim_{i\to\nu}(y_i, w_i)$ in $[\![\varphi]\!]$ is given by an ultra-arrow $r \colon x \rightsquigarrow \lim_{i\to\nu} y_i$ in X such that $r_\varphi(v) = (w_i)_{i\to\nu}$.

This assignment defines the *evaluation functor* $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$.

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Theorem

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Although we need X to be small to prove the above result, it follows easily that $\mathcal{E} \simeq \mathrm{Et}(\mathrm{pt}(\mathcal{E}))$. In logical terms, this reads as the following reconstruction result.

Theorem

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Let $\mathbb{T}$ be a geometric theory which is complete with respect to its **Set***-models. Then,* $\mathrm{Mod}(\mathbb{T})$ *is an ultraconvergence space by setting ultra-arrows $M \rightsquigarrow \lim_{i \to \nu} N_i$ to be structure morphisms $M \to \prod_{i \to \nu} N_i$, and* $\mathrm{Et}(\mathrm{Mod}(\mathbb{T}))$ *is the classifying topos of $\mathbb{T}$.*

The localic/propositional case

In particular, if a localic topos $\mathcal{E}$ has enough points, i.e. $\mathcal{E} \simeq \mathrm{Sh}(\mathcal{O}(X))$ for some topological space X, then $\mathcal{E} \simeq \mathrm{Et}(X)$.

Proof sketch

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1. $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is full on subobjects: every subobject of $\pi_\varphi \colon [\![\varphi]\!] \longrightarrow X$ in $\mathrm{Et}(X)$ is the restriction of π_φ to $[\![\psi]\!] \subseteq [\![\varphi]\!]$ for some subobject $\psi \rightarrowtail \varphi$ in $\mathcal{E}$.

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2. $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathsf{Et}(X)$ is covering: every étale space $p \colon Y \longrightarrow X$ is covered by an epimorphism $\alpha \colon \pi_\varphi \twoheadrightarrow p$ in $\mathsf{Et}(X)$ for some object $\varphi \in \mathcal{E}$.

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Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $[\![-]\!]$.

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Two points of view

Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $[\![-]\!]$. However, we can also interpret (1) as stating that $[\![-]\!]$ defines a hyperconnected geometric morphism, and (2) as stating that it defines a localic geometric morphism.

Ongoing work: ultraconvergence spaces as lax algebras

As it turns out, the inspiration from Barr's theorem can be pushed even further: in joint work in progress with Quentin Aristote, we have the following.

Theorem

The ultracompletion pseudomonad β *extends to a pseudomonad* $\underline{\beta}\colon \mathbf{PROF} \longrightarrow \mathbf{PROF}$*, and* $\mathbf{UltSp} \cong \mathrm{discLaxAlg}(\underline{\beta})$*.*

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This means that an ultraconvergence structure on a discrete category X can be equivalently specified by a profunctor $\Xi\colon \beta X \nrightarrow X$ and two transformations

$$
\begin{tikzcd}
X \arrow[rr, equal] \arrow[dr, "\eta_X"'] & \arrow[d, phantom, "\Downarrow \mathfrak{u}"] & X \\
& \beta X \arrow[ur, "\Xi"', "|" marking] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta^2 X \arrow[r, "\underline{\beta}\Xi", "|" marking] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\Swarrow \mathfrak{m}"] & \beta X \arrow[d, "\Xi", "|" marking] \\
\beta X \arrow[r, "\Xi"', "|" marking] & X
\end{tikzcd}
$$

satisfying appropriate axioms.

$$
\begin{tikzcd}
 & \mathbf{UltSp} \arrow[rr, leftrightarrow, "\cong" description] & & \mathsf{discLaxAlg}(\underline{\mathbb{\beta}}) \\
\mathbf{UltCat} \arrow[ur, hook] \arrow[rr, leftrightarrow, "\cong" description] & & \mathsf{PsAlg}(\mathbb{\beta}) \arrow[ur, hook] & \\
 & \mathbf{Top} \arrow[uu, hook] \arrow[rr, leftrightarrow, "\cong" description] & & \mathsf{LaxAlg}(\underline{\beta}) \arrow[uu, hook] \\
\mathbf{CompHaus} \arrow[uu, hook] \arrow[ur, hook] \arrow[rr, leftrightarrow, "\cong" description] & & \mathsf{Alg}(\beta) \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

Future work

- What is so fundamental about ultrafilters and ultraproducts in the reconstruction theorem? Can we drop the 'ultra' in 'ultraconvergence spaces', and obtain a more constructive version thereof dealing with filters and reduced products?

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- What is so fundamental about ultrafilters and ultraproducts in the reconstruction theorem? Can we drop the 'ultra' in 'ultraconvergence spaces', and obtain a more constructive version thereof dealing with filters and reduced products?
- Towards step (2) of our proof, we prove a kind of Beth definability theorem for geometric logic. What does this perspective entail?
- Can we describe the equivalences induced by the two adjunctions?

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Thank you!