Toposes with enough points as categories of étale spaces

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Compact Hausdorff spaces and convergence

Theorem (Manes)

CompHaus $\cong \mathrm{Alg}(\beta)$, *where* $\beta\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ *is the ultrafilter monad.*

This means that, for a compact Hausdorff space X, every function $f\colon I \longrightarrow X$ extends to a function $f^*\colon \beta I \longrightarrow X$ which we can think of as computing the *limit* of f with respect to each $\nu \in \beta I$. Concretely:

$$\begin{tikzcd} \beta I \arrow[dr, "f^*"] & \\ I \arrow[u, "\eta_I"] \arrow[r, "f"'] & X \end{tikzcd} \qquad f^*(\nu) = x \iff \forall U \subseteq X \text{ open, if } x \in U \text{ then } f^{-1}(U) \in \nu$$

In particular, the algebra map $\mathrm{id}_X^*\colon \beta X \longrightarrow X$ specifies, for each ultrafilter ν on X, the unique point of X all of whose open neighborhoods lie in ν.

Topological spaces and generalized convergence

For an arbitrary topological space X, these limits may not exist nor be unique, so that the previous definition of id_X^* determines a relation between βX and X.

Theorem (Barr)

*The ultrafilter monad β extends to a monad $\underline{\beta}\colon$ **Rel** $\longrightarrow$ **Rel**, and* **Top** $\cong \mathrm{LaxAlg}(\underline{\beta})$.

This means that a topology on a set X can be equivalently specified by a relation $\xi\colon \beta X \nrightarrow X$ such that:

$$
\begin{tikzcd}
X \arrow[rr, equal] \arrow[dr, "\eta_X"'] & \arrow[d, phantom, "\rotatebox[origin=c]{90}{$\subseteq$}"] & X \\
& \beta X \arrow[ur, "\xi"', "|" marking] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta^2 X \arrow[r, "\underline{\beta}\xi", "|" marking] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\supseteq"] & \beta X \arrow[d, "\xi", "-" marking] \\
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This means that a topology on a set X can be equivalently specified by a relation $\xi\colon X \times \beta X \to \mathbf{2}$ such that:

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$$

Notation

For $f\colon I \to X$ and $\nu \in \beta I$, we write $x \rightsquigarrow \lim_{i \to \nu} f(i)$ in case $\xi(x, \beta f(\nu))$ holds.

Now: one dimension higher!

Ultracategories and convergence of ultrafamilies

Going one dimension higher, the role of β is played by the *ultracompletion pseudomonad* $\upbeta\colon \mathbf{CAT} \longrightarrow \mathbf{CAT}$. For a category C, the category $\upbeta C$ has:

- as objects, triples (I, y, ν) of a set I, a functor $y\colon I \to C$, and an ultrafilter $\nu \in \beta I$;
- as morphisms $(I, y, \nu) \to (I', y', \nu')$, pairs of a function $h\colon I' \to I$ such that $\beta h(\nu') = \nu$ and a family of arrows $(\alpha_i\colon y_{h(i)} \to y'_i)_{i \in I'}$ in C, both considered up to ν'-equivalence.

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Intuitively, an ultracategory is a category C endowed with a functor $\Phi\colon \beta C \longrightarrow C$, assigning a unique *limit* in C to each *ultrafamily* (I, y, ν) in C. Formally, we define:

$$\mathbf{UltCat} := \mathrm{PsAlg}(\beta)$$

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Ultracategories categorify compact Hausdorff spaces

$\mathbf{CompHaus} \hookrightarrow \mathbf{UltCat}$ as those algebras whose carrier category is small and discrete.

Ultracategories and coherent theories

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- In the propositional case, such a reconstruction theorem is given by Stone duality. For a (classical) theory $\mathbb{T}$, the set $\mathrm{Mod}(\mathbb{T})$ admits a natural Stone topology such that $\mathbf{Stone}(\mathrm{Mod}(\mathbb{T}), \mathbf{2})$ is the (Boolean) Lindenbaum-Tarski algebra of $\mathbb{T}$.

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Theorem (Makkai; Lurie)

Let $\mathbb{T}$ be a coherent theory. Then, $\mathrm{Mod}(\mathbb{T})$ is an ultracategory by setting the limit of an ultrafamily (I, M_-, ν) of models to be their ultraproduct $\prod_{i \to \nu} M_i$, and $\mathbf{UltCat}(\mathrm{Mod}(\mathbb{T}), \mathbf{Set})$ is the classifying topos of $\mathbb{T}$.

Ultracategories and coherent toposes

Identifying coherent theories with coherent toposes, and restricting to the subcategory $\mathbf{UltCat}_*$ of ultracategories C such that $\mathbf{UltCat}(C, \mathbf{Set})$ is a topos, we have:

$$\begin{tikzcd} \mathbf{Topos}_{coh} \arrow[rr, hook, bend right, "\mathrm{pt}(-)"'] \arrow[rr, phantom, "\perp"] & & \mathbf{UltCat}_* \arrow[ll, bend right, "\mathbf{UltCat}(-{,}\mathbf{Set})"'] \end{tikzcd}$$

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$$\mathbf{Set}^I \xrightarrow{\prod_{i \to \nu}(-)} \mathbf{Set}$$

are *coherent*, i.e. they preserve finite limits, regular epimorphisms, and finite unions of subobjects.

What about geometric logic?

We will now consider geometric logic: a theory is geometric if its axioms are of the form $\forall \vec{x}(\varphi(\vec{x}) \to \psi(\vec{x}))$ where φ, ψ are built only using finitary $\wedge$, *infinitary* $\vee$, and $\exists$.

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A necessary restriction

Having such a result, for a geometric theory $\mathbb{T}$, entails its completeness with respect to its (**Set**-)models. Categorically, this corresponds to restricting to toposes with enough points, a condition analogue to *spatiality* for locales.

What about geometric logic?

$$\begin{tikzcd}
\mathbf{Topos}_{wep} \arrow[r, phantom, "\perp"] \arrow[r, hook, bend right, "\mathrm{pt}(-)"'] & {?} \arrow[l, dotted, bend right] \\
\mathbf{Topos}_{coh} \arrow[u, hook] \arrow[r, phantom, "\perp"] \arrow[r, hook, bend right, "\mathrm{pt}(-)"'] & \mathbf{UltCat} \arrow[u, hook] \arrow[l, dotted, bend right]
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\begin{tikzcd}
\mathbf{Topos}_{wep} \arrow[rr, hook, bend right, "\mathrm{pt}(-)"'] \arrow[rr, phantom, "\perp"] & & \mathbf{UltSp} \arrow[ll, dotted, bend right] \\
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$$\begin{tikzcd}[column sep=huge, row sep=huge]
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Ultraconvergence spaces

The key intuition to address this question comes from Barr's theorem: we can generalize ultracategories by replacing the algebra functor with a *profunctor*, and categorifying the description of topological spaces in terms of convergence.

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- for every $x \in X$, an *identity* ultra-arrow $\mathrm{id}_x\colon x \rightsquigarrow \lim_{* \to 1} x$;
- for every ultra-arrow $r\colon x \rightsquigarrow \lim_{i \to \mu} y_i$ and every ultrafamily of ultra-arrows $(s_i\colon y_i \rightsquigarrow \lim_{j \to \nu_i} z_{i,j})_{i \to \mu}$, a *composite* ultra-arrow $(s_i)_{i \to \mu} \cdot r\colon x \rightsquigarrow \lim_{(i,j) \to \sum_{i \to \mu} \nu_i} z_{i,j}$,

satisfying some equational axioms.

Continuous maps

Similarly, we can extend the notion of continuity to this **Set**-valued convergence relation, which now becomes *structure* rather than *property*.

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Definition

A continuous map of ultraconvergence spaces is a functor $f\colon X \longrightarrow X'$ together with a family of functions

$$\begin{aligned} \Xi(x,(I,y,\nu)) &\longrightarrow \Xi'(f(x),(I,fy,\nu)) \\ r\colon x \rightsquigarrow \lim_{i\to\nu} y_i &\longmapsto f(r)\colon f(x) \rightsquigarrow \lim_{i\to\nu} f(y_i) \end{aligned}$$

also satisfying some equational axioms.

With appropriate 2-cells, ultraconvergence spaces define a 2-category **UltSp**.

Examples

- Every topological space is a $\mathbf{2}$-valued ultraconvergence space with the usual convergence relation.

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$$\begin{tikzcd}
\mathbf{UltCat} \arrow[r, hook] & \mathbf{UltSp} \\
\mathbf{CompHaus} \arrow[r, hook] \arrow[u, hook] & \mathbf{Top} \arrow[u, hook]
\end{tikzcd}$$

The main theorem

As promised, the notion of ultraconvergence space allows us to obtain a reconstruction theorem for geometric logic: in topos-theoretical terms, it reads as follows.

Theorem (Saadia; Hamad; van Gool, Marquès, T.)

If $\mathcal{E}$ is a topos with enough points, then $\mathcal{E} \simeq \mathbf{UltSp}(\mathrm{pt}(\mathcal{E}), \mathbf{Set})$.

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In other words, restricting to the subcategory $\mathbf{UltSp}_*$ of ultraconvergence spaces X such that $\mathbf{UltSp}(X, \mathbf{Set})$ is a topos, we have what we wanted:

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Étale maps over B form a category $\mathrm{Et}(B)$, equivalent to $\mathbf{UltSp}(B, \mathbf{Set})$.

The ultraconvergence space of points of a topos

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- For every object $\varphi \in \mathcal{E}$, we can define an étale space $\pi_\varphi \colon [\![\varphi]\!] \longrightarrow X$ where:
  - the fiber of π_φ at $x \in X$ is given by $x(\varphi)$;
  - an ultra-arrow $(x, v) \rightsquigarrow \lim_{i \to \nu}(y_i, w_i)$ in $[\![\varphi]\!]$ is given by an ultra-arrow $r \colon x \rightsquigarrow \lim_{i \to \nu} y_i$ in X such that $r_\varphi(v) = (w_i)_{i \to \nu}$.

  This assignment defines the *evaluation functor* $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$.

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Theorem

If X is a separating set of points of $\mathcal{E}$, then $[\![-]\!] : \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is an equivalence.

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Theorem

Let $\mathbb{T}$ be a geometric theory which is complete with respect to its **Set***-models. Then, $\mathrm{Mod}(\mathbb{T})$ is an ultraconvergence space by setting ultra-arrows $M \rightsquigarrow \lim_{i\to\nu} N_i$ to be structure morphisms $M \to \prod_{i\to\nu} N_i$, and $\mathrm{Et}(\mathrm{Mod}(\mathbb{T}))$ is the classifying topos of $\mathbb{T}$.*

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The localic/propositional case

In particular, if a localic topos $\mathcal{E}$ has enough points, i.e. $\mathcal{E} \simeq \mathrm{Sh}(\mathcal{O}(X))$ for some topological space X, then $\mathcal{E} \simeq \mathrm{Et}(X)$.

Proof sketch

Our proof is substantially different from both Saadia's and Hamad's, who use Butz-Moerdijk's representation theorem for toposes with enough points. Instead, we proceed similarly to Makkai's original work, in two main steps.

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1. $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is full on subobjects: every subobject of $\pi_\varphi \colon [\![\varphi]\!] \longrightarrow X$ in $\mathrm{Et}(X)$ is the restriction of π_φ to $[\![\psi]\!] \subseteq [\![\varphi]\!]$ for some subobject $\psi \rightarrowtail \varphi$ in $\mathcal{E}$.

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2. $[\![-]\!]\colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is covering: every étale space $p\colon Y \longrightarrow X$ is covered by an epimorphism $\alpha\colon \pi_\varphi \twoheadrightarrow p$ in $\mathrm{Et}(X)$ for some object $\varphi \in \mathcal{E}$.

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Two points of view

Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $[\![-]\!]$. However, we can also interpret (1) as stating that $[\![-]\!]$ defines a hyperconnected geometric morphism, and (2) as stating that it defines a localic geometric morphism.

Ultraconvergence spaces as profunctorial algebras

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First, just as how the ultrafilter monad β extends to relations, we can extend the ultracompletion pseudomonad $\mathbb{\beta}$ to profunctors.

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The ultracompletion pseudomonad $\mathbb{\beta}$ extends to a pseudomonad $\underline{\mathbb{\beta}}\colon \mathbf{PROF} \longrightarrow \mathbf{PROF}$.

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The idea is that, as we can represent a relation $R\colon X \nrightarrow Y$ via the span $X \xleftarrow{\pi_X} R \xrightarrow{\pi_Y} Y$ of its two projection maps, we can identify a profunctor $F\colon C \nrightarrow D$ with a span $C \xleftarrow{\pi_C} R_F \xrightarrow{\pi_D} D$ of functors. This allows us to define $\underline{\mathbb{\beta}}$ by setting:

$$
\begin{tikzcd}
 & R_F \arrow[dl, "\pi_C"'] \arrow[dr, "\pi_D"] & & & & \mathbb{\beta} R_F \arrow[dl, "\mathbb{\beta}\pi_C"'] \arrow[dr, "\mathbb{\beta}\pi_D"] & \\
C & & D & \longmapsto & \mathbb{\beta} C & & \mathbb{\beta} D
\end{tikzcd}
$$

Ultraconvergence spaces as profunctorial algebras

Then, an ultraconvergence structure on a discrete category X can be equivalently specified by a profunctor $\Xi\colon \beta X \to\!\!\!\!\!\mapsto X$ and two transformations

$$
\begin{tikzcd}
X \arrow[rr, equal] \arrow[dr, "\eta_X"'] & \arrow[d, phantom, "\Downarrow\mathfrak{u}"] & X \\
& \beta X \arrow[ur, "\Xi"', "\shortmid"{marking}] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta^2 X \arrow[r, "\underline{\beta}\Xi", "\shortmid"{marking}] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\Swarrow\mathfrak{m}"] & \beta X \arrow[d, "\Xi", "\shortmid"{marking}] \\
\beta X \arrow[r, "\Xi"', "\shortmid"{marking}] & X
\end{tikzcd}
$$

satisfying the coherence axioms of a lax $\underline{\beta}$-algebra.

Theorem (Aristote, T.)

Ultraconvergence spaces coincide with the discrete lax $\underline{\beta}$-algebras.

$$
\begin{tikzcd}
 & \mathbf{UltSp} \arrow[rr, leftrightarrow, "\cong" description] & & \mathsf{discLaxAlg}(\underline{\mathbb{\beta}}) \\
\mathbf{UltCat} \arrow[ur, hook] \arrow[rr, leftrightarrow, "\cong" description] & & \mathsf{PsAlg}(\mathbb{\beta}) \arrow[ur, hook] & \\
 & \mathbf{Top} \arrow[uu, hook] \arrow[rr, leftrightarrow, "\cong" description] & & \mathsf{LaxAlg}(\underline{\beta}) \arrow[uu, hook] \\
\mathbf{CompHaus} \arrow[uu, hook] \arrow[ur, hook] \arrow[rr, leftrightarrow, "\cong" description] & & \mathsf{Alg}(\beta) \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

Future work

- What is so fundamental about ultrafilters and ultraproducts in the reconstruction theorem? Can we drop the 'ultra' in 'ultraconvergence spaces', and obtain a more constructive version thereof dealing with filters and reduced products?

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Thank you!

E. G. Manes (1969). "A triple theoretic construction of compact algebras". In: *Seminar on Triples and Categorical Homology Theory*. Ed. by B. Eckmann. Vol. 80. Lecture notes in Mathematics. Berlin, Heidelberg: Springer, pp. 91–118.

M. Barr (1970). "Relational algebras". In: *Reports of the Midwest Category Seminar IV*. Ed. by S. MacLane et al. Berlin, Heidelberg: Springer, pp. 39–55.

M. Makkai (1987). "Stone duality for first order logic". In: *Advances in Mathematics* 65.2, pp. 97–170.

J. Lurie (2018). *Ultracategories*. Available online.

R. Garner (2020). "Ultrafilters, finite coproducts and locally connected classifying toposes". In: *Annals of Pure and Applied Logic* 171.10, p. 102831.

G. Rosolini (2024). *Ultracompletions*. Talk at CT2024.

G. Saadia (2025). *Extending conceptual completeness via virtual ultracategories*. arXiv: 2506.23935.

A. Hamad (2025). *Generalised ultracategories and conceptual completeness of geometric logic*. arXiv: 2507.07922.

S. van Gool, J. Marquès, and U. Tarantino (2025). *Toposes with enough points as categories of étale spaces*. arXiv: 2508.09604.

Q. Aristote and U. Tarantino (2026). *Profunctorial algebras*. arXiv: 2601.22721.