

Toposes with enough points as categories of étale spaces

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Compact Hausdorff spaces and convergence

Theorem (Manes)

CompHaus $\cong \text{Alg}(\beta)$, where $\beta: \mathbf{Set} \rightarrow \mathbf{Set}$ is the ultrafilter monad.

This means that, for a compact Hausdorff space X , every function $f: I \rightarrow X$ extends to a function $f^*: \beta I \rightarrow X$ which we can think of as computing the *limit* of f with respect to each $\nu \in \beta I$. Concretely:

$$\begin{array}{ccc} \beta I & & \\ \eta \uparrow & \searrow f^* & \\ I & \xrightarrow{f} & X \end{array} \quad f^*(\nu) = x \iff \forall U \subseteq X \text{ open, if } x \in U \text{ then } f^{-1}(U) \in \nu$$

In particular, the algebra map $\text{id}_X^*: \beta X \rightarrow X$ specifies, for each ultrafilter ν on X , the unique point of X all of whose open neighborhoods lie in ν .

Topological spaces and generalized convergence

For an arbitrary topological space X , these limits may not exist nor be unique, so that the previous definition of id_X^* determines a relation between βX and X .

Theorem (Barr)

The ultrafilter monad β extends to a monad $\underline{\beta}: \mathbf{Rel} \rightarrow \mathbf{Rel}$, and $\mathbf{Top} \cong \mathbf{LaxAlg}(\underline{\beta})$.

This means that a topology on a set X can be equivalently specified by a relation $\xi: \beta X \rightarrow X$ such that:

$$\begin{array}{ccc}
 X & \xlongequal{\quad} & X \\
 \eta_X \searrow & \cap & \nearrow \xi \\
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 \beta^2 X & \xrightarrow{\quad \underline{\beta} \xi \quad} & \beta X & & \\
 \mu_X \downarrow & \supseteq & \downarrow \xi & & \\
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Notation

For $f: I \rightarrow X$ and $\nu \in \beta I$, we write $x \rightsquigarrow \lim_{i \rightarrow \nu} f(i)$ in case $\xi(x, \beta f(\nu))$ holds.

Now: one dimension higher!

Ultracategories and convergence of ultrafamilies

Going one dimension higher, the role of β is played by the *ultracompletion pseudomonad* $\beta: \mathbf{CAT} \longrightarrow \mathbf{CAT}$. For a category C , the category βC has:

- ▶ as objects, triples (I, y, ν) of a set I , a functor $y: I \rightarrow C$, and an ultrafilter $\nu \in \beta I$;
- ▶ as morphisms $(I, y, \nu) \rightarrow (I', y', \nu')$, pairs of a function $h: I' \rightarrow I$ such that $\beta h(\nu') = \nu$ and a family of arrows $(\alpha_i: y_{h(i)} \rightarrow y'_i)_{i \in I'}$ in C , both considered up to ν' -equivalence.

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Intuitively, an **ultracategory** is a category C endowed with a functor $\Phi: \beta C \longrightarrow C$, assigning a unique *limit* in C to each *ultrafamily* (I, y, ν) in C . Formally, we define:

$$\mathbf{UltCat} := \mathbf{PsAlg}(\beta)$$

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Ultracategories categorify compact Hausdorff spaces

CompHaus \hookrightarrow **UltCat** as those algebras whose carrier category is small and discrete.

Ultracategories and coherent theories

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For first-order logic, the role of Lindenbaum-Tarski algebras is played by **classifying toposes**: theories with equivalent classifying toposes have essentially the same models.

Theorem (Makkai; Lurie)

*Let \mathbb{T} be a coherent theory. Then, $\text{Mod}(\mathbb{T})$ is an ultracategory by setting the limit of an ultrafamily (I, M_-, ν) of models to be their ultraproduct $\prod_{i \rightarrow \nu} M_i$, and **UltCat**($\text{Mod}(\mathbb{T}), \mathbf{Set}$) is the classifying topos of \mathbb{T} .*

Ultracategories and coherent toposes

Identifying coherent theories with coherent toposes, and restricting to the subcategory \mathbf{UltCat}_* of ultracategories C such that $\mathbf{UltCat}(C, \mathbf{Set})$ is a topos, we have:

$$\begin{array}{ccc} & \mathbf{UltCat}(-, \mathbf{Set}) & \\ \swarrow & \perp & \searrow \\ \mathbf{Topos}_{coh} & & \mathbf{UltCat}_* \\ \nwarrow & \text{pt}(-) & \nearrow \end{array}$$

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This result crucially rests on **Łoś's theorem**: for a coherent theory, an ultraproduct of models is itself a model. Categorically, this is encoded by the fact that the ultraproduct functors

$$\mathbf{Set}' \xrightarrow{\prod_{i \rightarrow \nu} (-)} \mathbf{Set}$$

are *coherent*, i.e. they preserve finite limits, regular epimorphisms, and finite unions of subobjects.

What about geometric logic?

We will now consider **geometric logic**: a theory is geometric if its axioms are of the form $\forall \vec{x}(\varphi(\vec{x}) \rightarrow \psi(\vec{x}))$ where φ, ψ are built only using finitary \wedge , *infinitary* \vee , and \exists . For a geometric theory, Łoś's theorem fails: the ultraproduct functors are not *geometric*, as they don't preserve arbitrary unions of subobjects.

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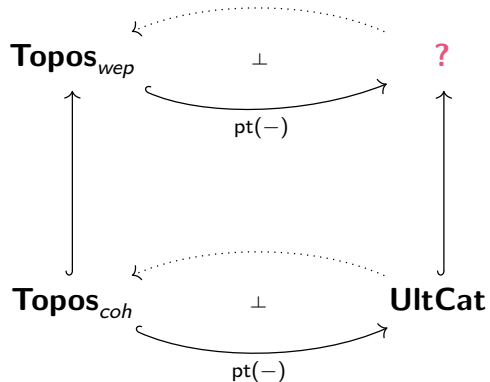
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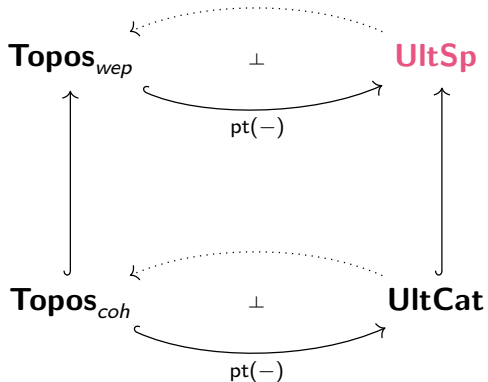
A necessary restriction

Having such a result, for a geometric theory \mathbb{T} , entails its completeness with respect to its (**Set**-)models. Categorically, this corresponds to restricting to toposes **with enough points**, a condition analogue to *spatiality* for locales.

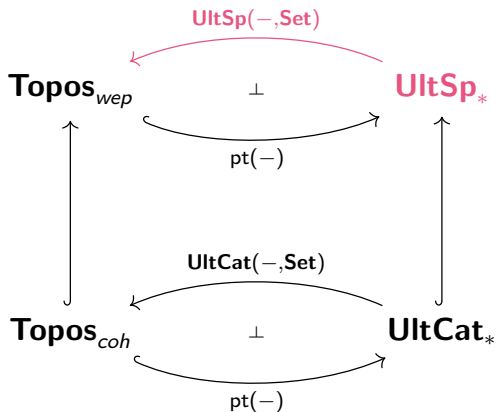
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- ▶ for every $x \in X$, an *identity* ultra-arrow $\text{id}_x: x \rightsquigarrow \lim_{* \rightarrow 1} x$;
- ▶ for every ultra-arrow $r: x \rightsquigarrow \lim_{i \rightarrow \mu} y_i$ and every ultrafamily of ultra-arrows $(s_i: y_i \rightsquigarrow \lim_{j \rightarrow \nu_i} z_{i,j})_{i \rightarrow \mu}$, a *composite* ultra-arrow $(s_i)_{i \rightarrow \mu} \cdot r: x \rightsquigarrow \lim_{(i,j) \rightarrow \sum_{i \rightarrow \mu} \nu_i} z_{i,j}$,

satisfying some equational axioms.

Continuous maps

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Definition

A **continuous map** of ultraconvergence spaces is a functor $f: X \longrightarrow X'$ together with a family of functions

$$\begin{aligned}\Xi(x, (I, y, \nu)) &\longrightarrow \Xi'(f(x), (I, fy, \nu)) \\ r: x \rightsquigarrow \lim_{i \rightarrow \nu} y_i &\longmapsto f(r): f(x) \rightsquigarrow \lim_{i \rightarrow \nu} f(y_i)\end{aligned}$$

also satisfying some equational axioms.

With appropriate 2-cells, ultraconvergence spaces define a 2-category **UltSp**.

Examples

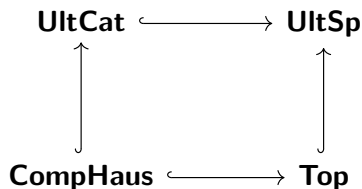
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The main theorem

As promised, the notion of ultraconvergence space allows us to obtain a reconstruction theorem for geometric logic: in topos-theoretical terms, it reads as follows.

Theorem (Saadia; Hamad; van Gool, Marquès, T.)

If \mathcal{E} is a topos with enough points, then $\mathcal{E} \simeq \mathbf{UltSp}(\mathrm{pt}(\mathcal{E}), \mathbf{Set})$.

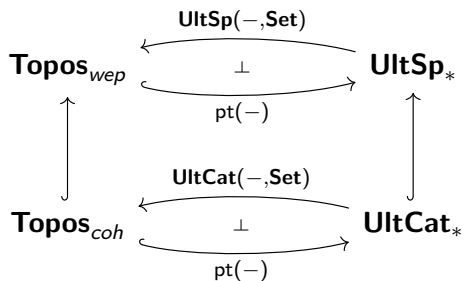
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In other words, restricting to the subcategory \mathbf{UltSp}_* of ultraconvergence spaces X such that $\mathbf{UltSp}(X, \mathbf{Set})$ is a topos, we have what we wanted:



Étale spaces

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Étale maps over B form a category $\text{Et}(B)$, equivalent to $\mathbf{UltSp}(B, \mathbf{Set})$.

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- ▶ For every object $\varphi \in \mathcal{E}$, we can define an étale space $\pi_\varphi: \llbracket \varphi \rrbracket \longrightarrow X$ where:
 - ▶ the fiber of π_φ at $x \in X$ is given by $x(\varphi)$;
 - ▶ an ultra-arrow $(x, \nu) \rightsquigarrow \lim_{i \rightarrow \nu} (y_i, w_i)$ in $\llbracket \varphi \rrbracket$ is given by an ultra-arrow $r: x \rightsquigarrow \lim_{i \rightarrow \nu} y_i$ in X such that $r_\varphi(\nu) = (w_i)_{i \rightarrow \nu}$.

This assignment defines the *evaluation functor* $\llbracket - \rrbracket: \mathcal{E} \longrightarrow \text{Et}(X)$.

Reconstruction for geometric logic

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Although we need X to be small to prove the above result, it follows easily that $\mathcal{E} \simeq \text{Et}(\text{pt}(\mathcal{E}))$. In logical terms, this reads as the following reconstruction result.

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*Let \mathbb{T} be a geometric theory which is complete with respect to its **Set**-models. Then, $\text{Mod}(\mathbb{T})$ is an ultraconvergence space by setting ultra-arrows $M \rightsquigarrow \lim_{i \rightarrow \nu} N_i$ to be structure morphisms $M \rightarrow \prod_{i \rightarrow \nu} N_i$, and $\text{Et}(\text{Mod}(\mathbb{T}))$ is the classifying topos of \mathbb{T} .*

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Although we need X to be small to prove the above result, it follows easily that $\mathcal{E} \simeq \text{Et}(\text{pt}(\mathcal{E}))$. In logical terms, this reads as the following reconstruction result.

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*Let \mathbb{T} be a geometric theory which is complete with respect to its **Set**-models. Then, $\text{Mod}(\mathbb{T})$ is an ultraconvergence space by setting ultra-arrows $M \rightsquigarrow \lim_{i \rightarrow \nu} N_i$ to be structure morphisms $M \rightarrow \prod_{i \rightarrow \nu} N_i$, and $\text{Et}(\text{Mod}(\mathbb{T}))$ is the classifying topos of \mathbb{T} .*

The localic/propositional case

In particular, if a localic topos \mathcal{E} has enough points, i.e. $\mathcal{E} \simeq \text{Sh}(\mathcal{O}(X))$ for some topological space X , then $\mathcal{E} \simeq \text{Et}(X)$.

Proof sketch

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1. $\llbracket - \rrbracket: \mathcal{E} \longrightarrow \text{Et}(X)$ is **full on subobjects**: every subobject of $\pi_\varphi: \llbracket \varphi \rrbracket \longrightarrow X$ in $\text{Et}(X)$ is the restriction of π_φ to $\llbracket \psi \rrbracket \subseteq \llbracket \varphi \rrbracket$ for some subobject $\psi \rightharpoonup \varphi$ in \mathcal{E} .

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2. $\llbracket - \rrbracket: \mathcal{E} \longrightarrow \text{Et}(X)$ is **covering**: every étale space $p: Y \longrightarrow X$ is covered by an epimorphism $\alpha: \pi_\varphi \twoheadrightarrow p$ in $\text{Et}(X)$ for some object $\varphi \in \mathcal{E}$.

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Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $\llbracket - \rrbracket$.

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Two points of view

Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $\llbracket - \rrbracket$. However, we can also interpret (1) as stating that $\llbracket - \rrbracket$ defines a hyperconnected geometric morphism, and (2) as stating that it defines a localic geometric morphism.

Ultraconvergence spaces as profunctorial algebras

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The ultracompletion pseudomonad $\mathbb{\beta}$ extends to a pseudomonad $\underline{\mathbb{\beta}}: \mathbf{PROF} \longrightarrow \mathbf{PROF}$.

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The idea is that, as we can represent a relation $R: X \rightarrowtail Y$ via the span $X \xleftarrow{\pi_X} R \xrightarrow{\pi_Y} Y$ of its two projection maps, we can identify a profunctor $F: C \rightarrowtail D$ with a span $C \xleftarrow{\pi_C} R_F \xrightarrow{\pi_D} D$ of functors. This allows us to define $\underline{\beta}$ by setting:

$$\begin{array}{ccc} & R_F & \\ \pi_C \swarrow & & \searrow \pi_D \\ C & & D \end{array} \quad \longmapsto \quad \begin{array}{ccc} & \beta R_F & \\ \beta \pi_C \swarrow & & \searrow \beta \pi_D \\ \beta C & & \beta D \end{array}$$

Ultraconvergence spaces as profunctorial algebras

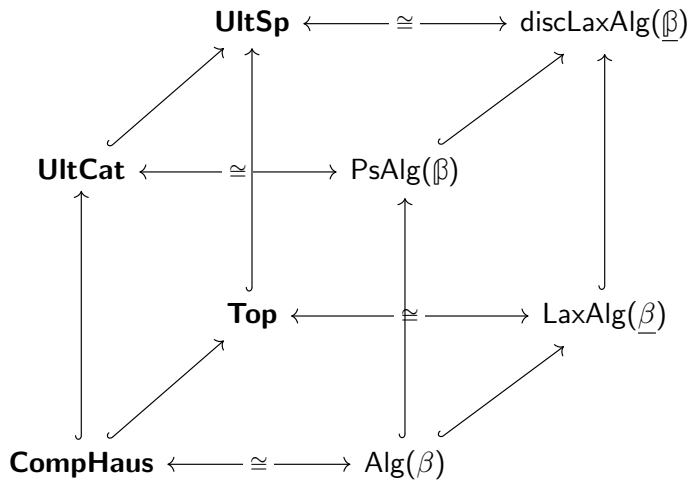
Then, an ultraconvergence structure on a discrete category X can be equivalently specified by a profunctor $\Xi: \beta X \rightarrowtail X$ and two transformations

$$\begin{array}{ccc}
 X & \xlongequal{\quad} & X \\
 \searrow \eta_X & \Downarrow u & \nearrow \Xi \\
 & \beta X &
 \end{array}
 \qquad
 \begin{array}{ccccc}
 \beta^2 X & \xrightarrow{\quad \beta \Xi \quad} & \beta X & & \\
 \mu_X \downarrow & \swarrow m & \downarrow \Xi & & \\
 \beta X & \xrightarrow{\quad \Xi \quad} & X & &
 \end{array}$$

satisfying the coherence axioms of a **lax β -algebra**.

Theorem (Aristote, T.)

Ultraconvergence spaces coincide with the discrete lax β -algebras.



Future work

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Thank you!

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