Arrow algebras, algebraic structures for modified realizability

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Arrow algebras

Arrow algebras

An arrow algebra $\mathcal{A}$ is a complete lattice $(A, \preccurlyeq)$ with an implication operator $\to\colon A^{\mathrm{op}} \times A \to A$ and a separator $S \subseteq A$ such that:

1. if $a \in S$ and $a \preccurlyeq b$, then $b \in S$;
2. if $a, a \to b \in S$, then $b \in S$;
3. S contains the following combinators:

$$\mathbf{k} \coloneqq \bigcurlywedge_{a,b} a \to b \to a$$

$$\mathbf{s} \coloneqq \bigcurlywedge_{a,b,c} (a \to b \to c) \to (a \to b) \to a \to c$$

$$\mathbf{a} \coloneqq \bigcurlywedge_{a,(b_i)_{i\in I},(c_i)_{i\in I}} \left(\bigcurlywedge_{i\in I} a \to b_i \to c_i \right) \to a \to \left(\bigcurlywedge_{i\in I} b_i \to c_i \right)$$

van den Berg, Briët, *Arrow algebras*, 2023

Arrow algebras

Elements of A should be thought of as truth values, or pieces of evidence; S singles out those that we consider "true", or conclusive.

Examples

1. Implicative algebras.
2. Frames, with the separator $\{\top\}$.
3. For a PCA $\mathbb{P} = (P, \leq, \cdot, P^{\#})$, the poset $(DP, \subseteq)$ with implication

$$\alpha \to \beta := \{\, c \in P \mid (\forall a \in \alpha)(c \cdot a\downarrow \text{ and } c \cdot a \in \beta \,\}$$

and separator $\left\{\, \alpha \in DP \;\middle|\; \exists r \in \alpha \cap P^{\#} \,\right\}$.

Arrow triposes and toposes

For any set I, functions $I \longrightarrow A$ can be preordered by:

$$\varphi \vdash_I \psi \iff \bigcurlywedge_{i \in I} \varphi(i) \to \psi(i) \in S$$

Theorem (van den Berg & Briët, 2023)

Every arrow algebra $\mathcal{A}$ induces a tripos:

$$P_{\mathcal{A}} : \mathrm{Set}^{\mathrm{op}} \longrightarrow \mathrm{HeytPre} \qquad \begin{array}{ccc} I & \longmapsto & (A^I, \vdash_I) \\ {\scriptstyle f}\uparrow & & \downarrow{\scriptstyle -\circ f} \\ J & \longmapsto & (A^J, \vdash_J) \end{array}$$

and hence a topos $\mathrm{AT}(\mathcal{A}) \coloneqq \mathrm{Set}[P_{\mathcal{A}}]$.

Examples

For a PCA $\mathbb{P}$, $\mathrm{AT}(DP)$ coincides with the realizability topos $\mathrm{RT}(\mathbb{P})$.

Morphisms of arrow algebras

Morphisms of arrow algebras

A morphism $f\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a function $f\colon A \longrightarrow B$ such that:

1. $f(S_A) \subseteq S_B$;
2. $$\bigcurlywedge_{a,a' \in A} f(a \to a') \to f(a) \to f(a') \in S_B;$$
3. for every $I \subseteq A \times A$,
$$\text{if} \bigcurlywedge_{(a,a') \in I} a \to a' \in S_A \text{ then } \bigcurlywedge_{(a,a') \in I} f(a) \to f(a') \in S_B.$$

Ordering morphisms $\mathcal{A} \longrightarrow \mathcal{B}$ as in $(B^A, \vdash_A)$ yields a preorder-enriched category ArrAlg.

T., *A category of arrow algebras for modified realizability*, 2025

Morphisms of arrow algebras

These morphisms correspond to cartesian transformations of the induced triposes:

$$\text{ArrAlg} \qquad\qquad \text{Tripos}_{\text{cart}}$$

$$\mathcal{A} \xrightarrow{\quad f \quad} \mathcal{B} \qquad \simeq \qquad P_{\mathcal{A}} \xrightarrow{\quad f \circ - \quad} P_{\mathcal{B}}$$

In particular, morphisms which are left adjoints in ArrAlg correspond to geometric transformations of the induced triposes.

$$\text{ArrAlg}_{\text{ladj}} \qquad\qquad \text{Tripos}_{\text{geom}}$$

$$\mathcal{A} \underset{h}{\overset{f}{\rightleftarrows}} \mathcal{B}\ (f \dashv h) \qquad \simeq \qquad P_{\mathcal{A}} \underset{h \circ -}{\overset{f \circ -}{\rightleftarrows}} P_{\mathcal{B}}\ (f \circ - \dashv h \circ -)$$

Morphisms of arrow algebras

Between frames

Let X, Y be frames seen as arrow algebras.

- Morphisms $X \longrightarrow Y$ in ArrAlg coincide with finite-meet preserving functions $X \longrightarrow Y$.
- Left adjoints $X \longrightarrow Y$ in ArrAlg coincide with frame homomorphisms $X \longrightarrow Y$.

Morphisms of arrow algebras

Between PCAs

Let $\mathbb{P}, \mathbb{Q}$ be PCAs. Note that $(DP, \subseteq)$ also carries the structure of a PCA, with application

$$\alpha \cdot \beta := \downarrow\{\, a \cdot b \mid a \in \alpha,\ b \in \beta \,\}$$

in case $a \cdot b\downarrow$ for all $a \in \alpha, b \in \beta$, and filter $(DP)^{\#} := S_{DP}$. This construction determines a pseudomonad on the preoder-enriched category of PCAs and morphisms of PCAs. A Kleisli morphism $\mathbb{P} \longrightarrow \mathbb{Q}$ is called a partial applicative morphism.

- Morphisms $DP \longrightarrow DQ$ in ArrAlg coincide with morphisms of PCAs $DP \longrightarrow DQ$.
- Left adjoints $DP \longrightarrow DQ$ in ArrAlg coincide with computationally dense partial applicative morphisms $\mathbb{P} \longrightarrow \mathbb{Q}$.

Nuclei and subtoposes

A nucleus on $\mathcal{A}$ is a morphism $j\colon \mathcal{A} \longrightarrow \mathcal{A}$ such that:

1. $\mathrm{id}_A \vdash_A j$, meaning $\curlywedge_a\, a \to ja \in S$;
2. $jj \vdash_A j$, meaning $\curlywedge_a\, jja \to ja \in S$.

A nucleus j induces a new arrow algebra $\mathcal{A}_j = (A, \preccurlyeq, \to_j, S_j)$ where:

$$a \to_j b \coloneqq a \to jb \qquad S_j \coloneqq \{\, a \in A \mid ja \in S \,\}$$

Proposition (T., 2025)

Every subtopos of $\mathrm{AT}(\mathcal{A})$ *is induced by an essentially unique nucleus j on $\mathcal{A}$ via the subtripos:*

$$P_{\mathcal{A}_j} \underset{j \circ -}{\overset{\mathrm{id}_A \circ -}{\rightleftarrows}} P_{\mathcal{A}} \qquad (\bot)$$

Arrow algebras for modified realizability

Modified realizability

At the level of PCAs, the key idea of Kreisel's modified realizability is to separate between a set of potential realizers always containing a canonical one, and a potentially empty subset of actual realizers. To construct the modified realizability topos Mod, Grayson defined a variant of the effective tripos having set of truth values:

$$\{\,(\alpha,\beta)\in\mathcal{P}\mathbb{N}\times\mathcal{P}\mathbb{N}\mid\alpha\subseteq\beta,\ 0\in\beta\,\}$$

where 0 is taken to be a code for the constant zero function.

Grayson, Modified realisability toposes, 1981

van Oosten, The modified realizability topos, 1997

Johnstone, Functoriality of modified realizability, 2017

Arrow algebras for modified realizability

For (almost) any arrow algebras $\mathcal{A}$, we can define a new arrow algebra $\mathcal{A}^{\to}$ on the set

$$A^{\to} := \{\, x = (x_0, x_1) \in A \times A \mid x_0 \preccurlyeq x_1 \,\}$$

with pointwise order, implication

$$x \to y := (x_0 \to y_0 \curlywedge x_1 \to y_1, x_1 \to y_1)$$

and separator

$$S^{\to} := \{\, x \in A^{\to} \mid x_0 \in S \,\}$$

Arrow algebras for modified realizability

On any arrow algebra $\mathcal{A}$, every element $u \in A$ defines the nuclei:

$$o(x) := u \to x \qquad c(x) := x \vee u$$

inducing respectively an open subtopos of $\mathsf{AT}(\mathcal{A})$ and its closed complement.

Proposition (T., 2025)

$\mathsf{AT}(\mathcal{A})$ is an open subtopos of $\mathsf{AT}(\mathcal{A}^{\to})$, induced by the nucleus $o(x) := (\bot, \top) \to x$.

Arrow algebras for modified realizability

We define the modification of $\mathcal{A}$ as the arrow algebra $\mathcal{A}^m := \mathcal{A}_c^{\rightarrow}$, where c is the closed nucleus on $\mathcal{A}^{\rightarrow}$ given by

$$c(x) := x \vee (\bot, \top)$$

so that $\mathsf{AT}(\mathcal{A}^m)$ is the closed complement of $\mathsf{AT}(\mathcal{A})$ as subtoposes of $\mathsf{AT}(\mathcal{A}^{\rightarrow})$.

Eff and Mod

Let $\mathcal{A} = D\mathcal{K}_1$. Then, $\mathsf{AT}(\mathcal{A}^m) \simeq \mathsf{Mod}$, so that we reobtain a result originally proved by van Oosten: Mod is the closed complement of $\mathsf{Eff} = \mathsf{AT}(\mathcal{A})$ as subtoposes of $\mathsf{Eff}_{\cdot\to\cdot} = \mathsf{AT}(\mathcal{A}^{\rightarrow})$, the effective topos over Set^2.

Arrow algebras for modified realizability

We can then lift a result originally proved by Johnstone to the setting of arrow algebras: modified realizability is functorial.

Proposition (T., 2025)

Both associations $\mathcal{A} \mapsto \mathcal{A}^{\rightarrow}$ and $\mathcal{A} \mapsto \mathcal{A}^{m}$ are (pseudo)functorial. In particular, every adjoint pair $f \dashv h \colon \mathcal{B} \longrightarrow \mathcal{A}$ induces pullback squares of toposes and geometric morphisms:

$$\begin{array}{ccc} \mathsf{AT}(\mathcal{B}) & \longrightarrow & \mathsf{AT}(\mathcal{A}) \\ \big\downarrow & & \big\downarrow \\ \mathsf{AT}(\mathcal{B}^{\rightarrow}) & \longrightarrow & \mathsf{AT}(\mathcal{A}^{\rightarrow}) \end{array} \qquad \begin{array}{ccc} \mathsf{AT}(\mathcal{B}^{m}) & \longrightarrow & \mathsf{AT}(\mathcal{A}^{m}) \\ \big\downarrow & & \big\downarrow \\ \mathsf{AT}(\mathcal{B}^{\rightarrow}) & \longrightarrow & \mathsf{AT}(\mathcal{A}^{\rightarrow}) \end{array}$$

Arrow algebras for modified realizability

Remark

Mod models *Troelstra's variant* of Kreisel's modified realizability, based on the HRO model of $\mathbf{HA}^\omega$ instead of the HEO model. In particular, it doesn't validate the axiom of choice for finite types, characteristic of Kreisel's original notion.

For a PCA $\mathbb{P}$, we can endow the set $\mathrm{PER}(P)$ of partial equivalence relations on P with the structure of an arrow algebra. For $\mathbb{P} = \mathcal{K}_1$ and $\mathcal{A} = \mathrm{PER}(P)$, de Vries defined a subtopos of $\mathrm{AT}(\mathcal{A}^m)$ where AC holds for finite types: we still don't know whether it holds already in $\mathrm{AT}(\mathcal{A}^m)$.

de Vries, *An extensional modified realizability topos*, 2017

Thank you!