Arrow algebras, algebraic structures for modified realizability

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Arrow algebras

Arrow algebras

An arrow algebra \mathcal{A} is a complete lattice (A, \preccurlyeq) with an implication operator $\rightarrow: \mathcal{A}^{op} \times A \rightarrow A$ and a separator $S \subseteq A$ such that:

1. if
$$a \in S$$
 and $a \preccurlyeq b$, then $b \in S$;

2. if $a, a \rightarrow b \in S$, then $b \in S$;

3. S contains the following combinators:

$$\mathbf{k} := \bigwedge_{a,b} a \to b \to a$$
$$\mathbf{s} := \bigwedge_{a,b,c} (a \to b \to c) \to (a \to b) \to a \to c$$
$$\mathbf{a} := \bigwedge_{a,(b_i)_{i \in I}, (c_i)_{i \in I}} \left(\bigwedge_{i \in I} a \to b_i \to c_i \right) \to a \to \left(\bigwedge_{i \in I} b_i \to c_i \right)$$

van den Berg, Briët, Arrow algebras, 2023

Arrow algebras

Elements of *A* should be thought of as truth values, or pieces of evidence; *S* singles out those that we consider "true", or conclusive.

Examples

- 1. Implicative algebras.
- 2. Frames, with the separator $\{\top\}$.
- 3. For a PCA $\mathbb{P} = (P, \leq, \cdot, P^{\#})$, the poset (DP, \subseteq) with implication

$$\alpha \to \beta := \{ c \in P \mid (\forall a \in \alpha)(c \cdot a \downarrow \text{ and } c \cdot a \in \beta \}$$

and separator $\{ \alpha \in DP \mid \exists r \in \alpha \cap P^{\#} \}.$

Arrow triposes and toposes

For any set *I*, functions $I \longrightarrow A$ can be preordered by:

$$\varphi \vdash_I \psi \iff \bigwedge_{i \in I} \varphi(i) \to \psi(i) \in S$$

Theorem (van den Berg & Briët, 2023) Every arrow algebra A induces a tripos:

$$P_{\mathcal{A}}: \mathsf{Set}^{\mathsf{op}} \longrightarrow \mathsf{HeytPre} \qquad \begin{array}{c} I \longmapsto (A', \vdash_{I}) \\ f \uparrow \qquad \qquad \downarrow_{-\circ f} \\ J \longmapsto (A^{J}, \vdash_{J}) \end{array}$$

and hence a topos $AT(\mathcal{A}) \coloneqq Set[P_{\mathcal{A}}]$.

Examples

For a PCA \mathbb{P} , AT(*DP*) coincides with the realizability topos RT(\mathbb{P}).

A morphism $f: \mathcal{A} \longrightarrow \mathcal{B}$ is a function $f: \mathcal{A} \longrightarrow \mathcal{B}$ such that: 1. $f(S_A) \subseteq S_B$; 2.

$$\bigwedge_{a,a'\in A} f(a\to a') \to f(a) \to f(a') \in S_B;$$

3. for every $I \subseteq A \times A$,

$$\text{if } \bigwedge_{(a,a')\in I} a \to a' \in S_A \text{ then } \bigwedge_{(a,a')\in I} f(a) \to f(a') \in S_B.$$

Ordering morphisms $\mathcal{A} \longrightarrow \mathcal{B}$ as in $(\mathcal{B}^A, \vdash_A)$ yields a preorder-enriched category ArrAlg.

T., A category of arrow algebras for modified realizability, 2025

These morphisms correspond to cartesian transformations of the induced triposes:



In particular, morphisms which are left adjoints in ArrAlg correspond to geometric transformations of the induced triposes.



Between frames

Let X, Y be frames seen as arrow algebras.

- ► Morphisms X → Y in ArrAlg coincide with finite-meet preserving functions X → Y.
- ► Left adjoints X → Y in ArrAlg coincide with frame homomorphisms X → Y.

Between PCAs

Let \mathbb{P}, \mathbb{Q} be PCAs. Note that (DP, \subseteq) also carries the structure of a PCA, with application

$$\alpha \cdot \beta := \downarrow \{ a \cdot b \mid a \in \alpha, b \in \beta \}$$

in case $a \cdot b \downarrow$ for all $a \in \alpha, b \in \beta$, and filter $(DP)^{\#} \coloneqq S_{DP}$. This construction determines a pseudomonad on the preoder-enriched category of PCAs and morphisms of PCAs. A Kleisli morphism $\mathbb{P} \longrightarrow \mathbb{Q}$ is called a partial applicative morphism.

- ► Morphisms DP → DQ in ArrAlg coincide with morphisms of PCAs DP → DQ.
- ► Left adjoints DP → DQ in ArrAlg coincide with computationally dense partial applicative morphisms P → Q.

Nuclei and subtoposes

A nucleus on \mathcal{A} is a morphism $j: \mathcal{A} \longrightarrow \mathcal{A}$ such that:

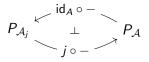
- 1. id_A $\vdash_A j$, meaning $\bigwedge_a a \rightarrow ja \in S$;
- 2. $jj \vdash_A j$, meaning $\bigwedge_a jja \rightarrow ja \in S$.

A nucleus *j* induces a new arrow algebra $\mathcal{A}_j = (A, \preccurlyeq, \rightarrow_j, S_j)$ where:

$$a \rightarrow_j b := a \rightarrow jb$$
 $S_j := \{ a \in A \mid ja \in S \}$

Proposition (T., 2025)

Every subtopos of AT(A) is induced by an essentially unique nucleus j on A via the subtripos:



Modified realizability

At the level of PCAs, the key idea of Kreisel's modified realizability is to separate between a set of potential realizers always containing a canonical one, and a potentially empty subset of actual realizers. To construct the modified realizability topos Mod, Grayson defined a variant of the effective tripos having set of truth values:

$$\{ (\alpha, \beta) \in \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N} \mid \alpha \subseteq \beta, \ \mathbf{0} \in \beta \}$$

where 0 is taken to be a code for the constant zero function.

Grayson, Modified realisability toposes, 1981 van Oosten, The modified realizability topos, 1997 Johnstone, Functoriality of modified realizability, 2017

For (almost) any arrow algebras $\mathcal A,$ we can define a new arrow algebra $\mathcal A^\to$ on the set

$$A^{\rightarrow} \coloneqq \{ x = (x_0, x_1) \in A \times A \mid x_0 \preccurlyeq x_1 \}$$

with pointwise order, implication

$$x \to y := (x_0 \to y_0 \not \downarrow x_1 \to y_1, x_1 \to y_1)$$

and separator

$$S^{\rightarrow} \coloneqq \{ x \in A^{\rightarrow} \mid x_0 \in S \}$$

On any arrow algebra A, every element $u \in A$ defines the nuclei:

$$o(x) \coloneqq u \to x$$
 $c(x) \coloneqq x \lor u$

inducing respectively an open subtopos of AT(A) and its closed complement.

Proposition (T., 2025)

AT(A) is an open subtopos of $AT(A^{\rightarrow})$, induced by the nucleus $o(x) := (\bot, \top) \rightarrow x$.

We define the modification of \mathcal{A} as the arrow algebra $\mathcal{A}^m := \mathcal{A}_c^{\rightarrow}$, where c is the closed nucleus on $\mathcal{A}^{\rightarrow}$ given by

$$c(x) \coloneqq x \lor (\bot, \top)$$

so that $AT(\mathcal{A}^m)$ is the closed complement of $AT(\mathcal{A})$ as subtoposes of $AT(\mathcal{A}^{\rightarrow})$.

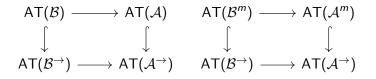
Eff and Mod

Let $\mathcal{A} = D\mathcal{K}_1$. Then, $\mathsf{AT}(\mathcal{A}^m) \simeq \mathsf{Mod}$, so that we reobtain a result originally proved by van Oosten: Mod is the closed complement of $\mathsf{Eff} = \mathsf{AT}(\mathcal{A})$ as subtoposes of $\mathsf{Eff}_{.\to.} = \mathsf{AT}(\mathcal{A}^{\to})$, the effective topos over Set^2 .

We can then lift a result originally proved by Johnstone to the setting of arrow algebras: modified realizability is functorial.

Proposition (T., 2025)

Both associations $\mathcal{A} \mapsto \mathcal{A}^{\rightarrow}$ and $\mathcal{A} \mapsto \mathcal{A}^{m}$ are (pseudo)functorial. In particular, every adjoint pair $f \dashv h: \mathcal{B} \longrightarrow \mathcal{A}$ induces pullback squares of toposes and geometric morphisms:



Remark

Mod models *Troelstra's variant* of Kreisel's modified realizability, based on the HRO model of \mathbf{HA}^{ω} instead of the HEO model. In particular, it doesn't validate the axiom of choice for finite types, characteristic of Kreisel's original notion.

For a PCA \mathbb{P} , we can endow the set PER(P) of partial equivalence relations on P with the structure of an arrow algebra. For $\mathbb{P} = \mathcal{K}_1$ and $\mathcal{A} = PER(P)$, de Vries defined a subtopos of $AT(\mathcal{A}^m)$ where AC holds for finite types: we still don't know whether it holds already in $AT(\mathcal{A}^m)$.

de Vries, An extensional modified realizability topos, 2017

Thank you!