Conceptual completeness for geometric logic via ultraconvergence spaces

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Warm up: the propositional classical case

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formulas modulo $\mathbb{T}$-provable equivalence form a Boolean algebra

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Each formula φ identifies the set of models in which it is true:

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$$[\![-]\!]\colon \mathsf{L}(\mathbb{T}) \longrightarrow 2^{\mathsf{Mod}(\mathbb{T})} \qquad [\![\varphi]\!](v) := v(\varphi)$$

- By the usual completeness theorem, $[\![-]\!]$ is injective.

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This isomorphism is a conceptual completeness theorem for propositional classical logic:

- a logic enjoys *conceptual completeness* if the syntax of a theory $\mathbb{T}$ in the logic can be reconstructed from its semantics $\mathrm{Mod}(\mathbb{T})$, provided that the latter is equipped with appropriate additional structure.

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first-order geometric $\wedge,\ \bigvee,\ \exists$ + sequents	ultraconvergence structure	[Saa25, Ham25, GMT26]

Topological spaces in terms of convergence

Fix a topological space X. An ultrafamily in X, i.e. an element $(x_i)_{i \to \mu}$ of some ultrapower $\prod_{i \to \mu} X$ of X, converges to some $x \in X$ if

$$\forall\ U \subseteq X \text{ open} : x \in U \Rightarrow \{\, i \in I \mid x_i \in U \,\} \in \mu,$$

in which case we write $x \rightsquigarrow \lim_{i \to \mu} x_i$.

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Relational β-algebras

The above relation restricts to one between X and the set βX of ultrafilters on X, which recovers Barr's description of topological spaces as *relational β-algebras* [Bar70].

Topological spaces in terms of convergence

This convergence relation satisfies the following properties:

1. $x \rightsquigarrow \lim_{* \to 1} x$;
   - 1 is the unique ultrafilter on the singleton $\{*\}$
2. if $x \rightsquigarrow \lim_{i \to \mu} y_i$ and $(y_i \rightsquigarrow \lim_{j \to \nu_i} z_{i,j})_{i \to \mu}$, then $x \rightsquigarrow \lim_{(i,j) \to \sum_{i \to \mu} \nu_i} z_{i,j}$;
   - if each ν_i is an ultrafilter on some set J_i, then $\sum_{i \to \mu} \nu_i$ is the ultrafilter on $\sum_{i \in I} J_i$ defined by the sets $\sum_{i \in U} V_i$ for $U \in \mu$ and $V_i \in \nu_i$
3. if $x \rightsquigarrow \lim_{j \to f_* \mu} y_j$ for some function $f : I \to J$, then $x \rightsquigarrow \lim_{i \to \mu} y_{f(i)}$.
   - $f_* \mu$ is the ultrafilter on J defined by the sets $V \subseteq J$ such that $f^{-1}(V) \in \mu$

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One dimension higher

Our ultraconvergence spaces arise as a categorification of such a relation, with a (possibly empty) set of 'witnesses of convergence' of each ultrafamily to each point.

Ultraconvergence spaces

Definition

An ultraconvergence space consists of a class X of *points* together with, for each point x and each ultrafamily of points $(y_i)_{i \to \mu}$, a set of *ultra-arrows* $r \colon x \rightsquigarrow \lim_{i \to \mu} y_i$.

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Moreover, X is endowed with:

1. for any point x, an *identity*
$$\mathrm{id}_x\colon x \rightsquigarrow \lim_{*\to 1} x;$$
2. for any ultra-arrow $r\colon x \rightsquigarrow \lim_{i\to\mu} y_i$ and any ultrafamily of ultra-arrows $(s_i\colon y_i \rightsquigarrow \lim_{j\to\nu_i} z_{i,j})_{i\to\mu}$, a *composition*
$$(s_i)_{i\to\mu}\cdot r\colon x \rightsquigarrow \lim_{(i,j)\to\sum_{i\to\mu}\nu_i} z_{i,j};$$
3. for any ultra-arrow $r\colon x \rightsquigarrow \lim_{j\to f_*\mu} y_j$ for some function $f\colon I\to J$, a *reindexing*
$$r[f]\colon x \rightsquigarrow \lim_{i\to\mu} y_{f(i)},$$

satisfying some equational axioms.

Continuous maps of ultraconvergence spaces

Definition

A continuous map $F\colon X \longrightarrow Y$ of ultraconvergence spaces assigns

1. a point $F(x)$ in Y to each point x in X, and
2. an ultra-arrow $F(r)\colon F(x) \rightsquigarrow \lim_{i\to\mu} F(x_i)$ in Y to each ultra-arrow $r\colon x \rightsquigarrow \lim_{i\to\mu} x_i$ in X,

preserving identities, compositions, and reindexings. With an appropriate notion of morphism, continuous maps $X \longrightarrow Y$ form a category $\mathrm{Cont}(X, Y)$.

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The topological case

If X and Y are topological spaces seen as ultraconvergence spaces, then $\mathrm{Cont}(X, Y)$ is the usual set of continuous maps $X \longrightarrow Y$ with the pointwise specialization order.

Spaces of models

Just as in the propositional case, we can thus endow the class $\mathrm{Mod}(\mathbb{T})$ of models of a geometric theory $\mathbb{T}$ in a signature Σ with an ultraconvergence structure: an ultra-arrow $M \rightsquigarrow \lim_{i \to \mu} N_i$ is a Σ-*structure homomorphism*

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Every formula $\varphi(x_1,\ldots,x_k)$ in the language Σ then induces its semantic evaluation

$$[\![\varphi]\!] \colon \mathrm{Mod}(\mathbb{T}) \longrightarrow \mathrm{Set} \qquad M \mapsto \{\, (m_1,\ldots,m_k) \in M^k \mid M \models \varphi(m_1,\ldots,m_k)\,\}$$

which can be naturally extended to a continuous map.

Classifying toposes

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Intuitively, toposes admit the categorical structure necessary to interpret geometric logic, and geometric morphisms preserve these interpretations.

Every geometric theory $\mathbb{T}$ admits a classifying topos $\mathsf{Set}[\mathbb{T}]$, built out of its syntax and characterized by the existence of a *generic* $\mathbb{T}$*-model*, yielding a correspondence

$$\{\, \text{geometric morphisms } \mathcal{E} \longrightarrow \mathsf{Set}[\mathbb{T}] \,\} \longleftrightarrow \{\, \text{models of } \mathbb{T} \text{ in } \mathcal{E} \,\}\,.$$

for any topos $\mathcal{E}$. Theories with equivalent classifying toposes essentially have the same models in any topos, so in particular in Set.

The reconstruction theorem

The assignment $\varphi \mapsto [\![\varphi]\!]$ extends to a functor

$$[\![-]\!] \colon \mathsf{Set}[\mathbb{T}] \longrightarrow \mathsf{Cont}(\mathsf{Mod}(\mathbb{T}), \mathsf{Set}),$$

so that our main result can be stated as follows.

Theorem (Saadia; Hamad; van Gool, Marquès, T.)

For a geometric theory $\mathbb{T}$ *enjoying completeness with respect to its models in* Set*, the functor* $[\![-]\!] \colon \mathsf{Set}[\mathbb{T}] \longrightarrow \mathsf{Cont}(\mathsf{Mod}(\mathbb{T}), \mathsf{Set})$ *is an equivalence of categories.*

In topos-theoretic terms

Exploiting the fact that

1. every topos $\mathcal{E}$ classifies some geometric theory $\mathbb{T}$, and
2. in that case, points of $\mathcal{E}$ correspond to models of $\mathbb{T}$ (in Set),

our result can be rephrased as a representation theorem for toposes with enough points, a condition analogous to the one for frames.

Theorem

Every topos with enough points $\mathcal{E}$ is equivalent to the category $\mathrm{Cont}(\mathrm{pt}(\mathcal{E}), \mathrm{Set})$ *of continuous maps from its ultraconvergence space of points* $\mathrm{pt}(\mathcal{E})$ *to* Set.

Ongoing and future work

- A topology on a set X yields a more general convergence relation between points of X and filters on X, which is uniquely defined by its restriction to ultrafilters. Similarly, it is evident how to define a variant of ultraconvergence spaces in terms of filters and reduced products in place of ultrafilters and ultraproducts.
  1. How do the two notions compare? Is it still true that ultrafilters suffice to recover the whole convergence structure?
  2. What is so fundamental about ultrafilters and ultraproducts in the reconstruction theorem? Can we obtain a more constructive version thereof, dealing with filters?

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  1. How do the two notions compare? Is it still true that ultrafilters suffice to recover the whole convergence structure?
  2. What is so fundamental about ultrafilters and ultraproducts in the reconstruction theorem? Can we obtain a more constructive version thereof, dealing with filters?
- In the propositional case, spaces of models of geometric theories are precisely the *sober spaces*.
  3. Can we similarly characterize ultraconvergence spaces of the form $\mathrm{Mod}(\mathbb{T})$ for some geometric theory $\mathbb{T}$?

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Thank you!