An arrow-algebraic approach to realizability

Umberto Tarantino

IRIF, Université Paris Cité

CHoCoLa

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Introduction

Toposes as mathematical universes

A topos is a categorical setting for intuitionistic higher-order logic.

Logical language	**Interpretation in a topos $\mathcal{E}$**
types A	objects A
terms $x : A \vdash t(x) : B$	morphisms $A \longrightarrow B$
predicates $x : A \vdash \varphi(x) : \mathrm{Prop}$	subobjects of A
function types $A \to B$	exponentials B^A
type of propositions Prop	subobject classifier Ω

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In a topos $\mathcal{E}$, predicates on A correspond to morphisms $A \longrightarrow \Omega$ and to global elements of Ω^A, so they can be quantified over:

$$\mathrm{Sub}_{\mathcal{E}}(A) \cong \mathcal{E}(A, \Omega) \cong \mathcal{E}(1, \Omega^A)$$

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- predicates of type I are subsets of I, or functions $I \longrightarrow 2$;

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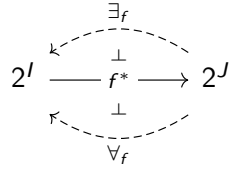
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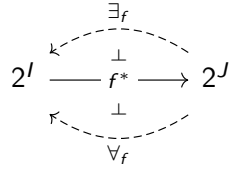
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- every predicate arises as the substitution of the generic predicate $\mathrm{id}_2 \in 2^2$:

$$\phi = \phi^*(\mathrm{id}_2) \in 2^I$$

Triposes

It turns out that these logical features are basically sufficient to determine a topos. Abstractly, the necessary structure can be axiomatized in the notion of a tripos.

Definition (Hyland, Johnstone and Pitts)

A tripos is a pseudofunctor

$$T : \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{Preord}$$

such that:

1. for any set I the preorder $T(I)$ is a Heyting prealgebra of predicates, and for any function $f : I \longrightarrow J$ the map $f^* = T(f) : T(J) \longrightarrow T(I)$ preserves the Heyting structure;
2. every map f^* has adjoints $\exists_f \dashv f^* \dashv \forall_f$ satisfying the Beck–Chevalley condition;
3. there is a generic predicate $\sigma \in T(\Sigma)$, i.e. such that every $\phi \in T(I)$ arises, up to isomorphism, as $u^*\sigma$ for some $u : I \longrightarrow \Sigma$.

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An object of $\mathbf{Set}[T]$ is a set I with a T-valued partial equivalence relation $\rho \in T(I \times I)$:

$$\rho(x,y) \vdash_{I\times I} \rho(y,x), \qquad \rho(x,y) \wedge \rho(y,z) \vdash_{I\times I\times I} \rho(x,z).$$

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Localic triposes and toposes

For a frame X, i.e. a complete Heyting algebra, the functor $I \longmapsto (X^I, \leq)$ is a tripos; substitution is by precomposition, and a generic predicate is $\mathrm{id}_X \in X^X$. The topos $\mathbf{Set}[X^-]$ is equivalent to the topos of sheaves $\mathrm{Sh}(X)$: in particular, $\mathbf{Set}[2^-] \simeq \mathbf{Set}$.

Realizability triposes and toposes

Fix a (relative) PCA $(P, \cdot, P^{\#})$, i.e. a set of realizers endowed with:

- an application $\cdot$, a partial binary operation;
- a filter $P^{\#} \subseteq P$, a subset of admissible realizers closed under defined application and containing suitable combinators $\mathbf{k}, \mathbf{s}$.

For a set I, predicates are functions $I \longrightarrow \mathrm{Pow}\, P$, ordered by:

$$\phi \vdash_I \psi \quad \Longleftrightarrow \quad \exists r \in P^{\#}\ \forall i \in I\ \forall a \in \phi_i : r \cdot a \downarrow \wedge\ r \cdot a \in \psi_i.$$

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With substitution by precomposition, $I \longmapsto ((\mathrm{Pow}\, P)^I, \vdash_I)$ defines the realizability tripos of P. The topos $\mathrm{RT}(P) = \mathbf{Set}[(\mathrm{Pow}\, P)^-]$ is then the realizability topos of P.

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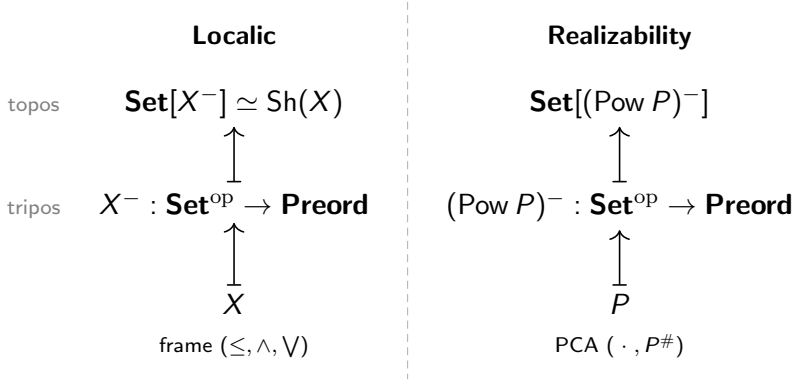
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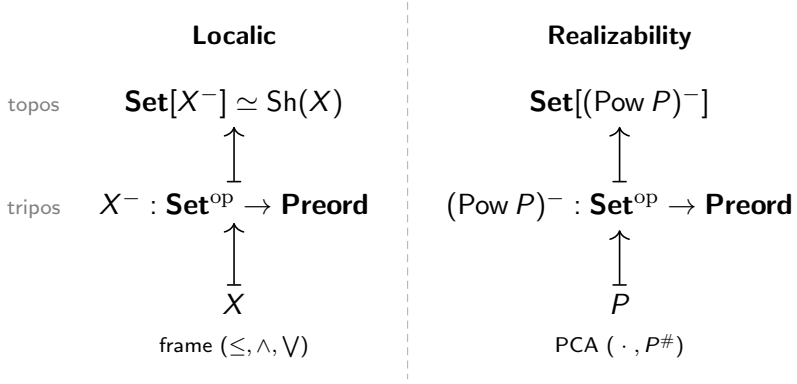
The archetypal example

Kleene's first algebra $\mathcal{K}_1$ is the PCA defined on $\mathbb{N}$ by partial recursive functions, i.e. $n \cdot m = \varphi_n(m)$ whenever defined, and $\mathrm{RT}(\mathcal{K}_1)$ is Hyland's effective topos **Eff**. Its internal logic is Kleene's realizability interpretation of intuitionistic logic.

Overview


Localic
Realizability
topos
tripos
$\mathbf{Set}[X^-] \simeq \mathrm{Sh}(X)$
$X^- : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Preord}$
X
frame $(\leq, \wedge, \bigvee)$
$\mathbf{Set}[(\mathrm{Pow}\, P)^-]$
$(\mathrm{Pow}\, P)^- : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Preord}$
P
PCA $(\,\cdot\,, P^{\#})$

Overview



These two classes of toposes are essentially disjoint, in that they intersect only in **Set**. Yet, both arise from triposes, built in turn from quite different algebraic structures: their internal logic can then be reduced to simple computations inside these structures.

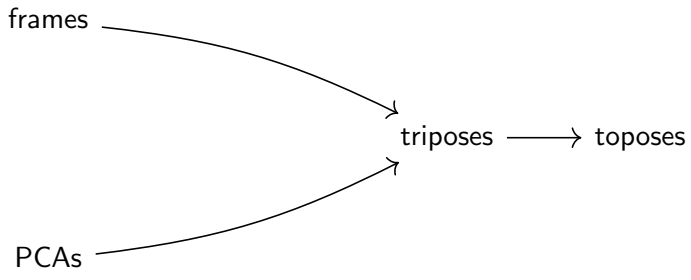
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Here we shall see arrow algebras, introduced by Benno van den Berg and Marcus Briët.



Arrow algebras

Truth values and evidence

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An arrow algebra $\mathcal{A}$ is defined by a tuple $(A, \preccurlyeq, \to, S)$ where:

- elements of A represent truth values, or pieces of evidence;
- the evidential order $a \preccurlyeq b$ expresses that b provides less evidence than a;
- the implication $a \to b$ represents evidence for the transformation of a into b;
- the separator $S \subseteq A$ singles out evidence that we consider conclusive.

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Combining implication and the separator we get the logical order on A, weaker than $\preccurlyeq$:

$$a \vdash b \quad \iff \quad a \to b \in S.$$

Arrow algebras

Definition (van den Berg and Briët)

An arrow algebra is a complete lattice $(A, \preccurlyeq)$ with a monotone operation $\to\colon A^{\mathrm{op}} \times A \to A$ and a subset $S \subseteq A$ such that:

1. $a \in S$, $a \preccurlyeq b$ imply $b \in S$;
2. $a \in S$, $a \to b \in S$ imply $b \in S$;
3. $\mathbf{k}, \mathbf{s}, \mathbf{a} \in S$, where:

$$\mathbf{k} = \bigwedge_{a,b \in A} a \to b \to a,$$

$$\mathbf{s} = \bigwedge_{a,b,c \in A} (a \to b \to c) \to (a \to b) \to a \to c,$$

$$\mathbf{a} = \bigwedge_{\substack{a \in A \\ B \subseteq \mathrm{im}(\to)}} \left(\bigwedge_{b \in B} a \to b \right) \to a \to \left(\bigwedge_{b \in B} b \right).$$

Frames and implicative algebras

Frames

A frame X is an arrow algebra with its Heyting implication and separator $S = \{\top\}$:

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Miquel's implicative algebras

By design, arrow algebras generalize Alexandre Miquel's implicative algebras, characterized by the stronger distributivity law

$$\bigwedge_{b \in B} (a \to b) = a \to \bigwedge_{b \in B} b$$

for any $B \subseteq A$, and quasi-implicative algebras by the same law for nonempty B.

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$$\alpha \to \beta = \{r \in P \mid \forall a \in \alpha : r \cdot a \downarrow \wedge\ r \cdot a \in \beta\},$$

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Arrow triposes

For an arrow algebra $\mathcal{A}$, we can preorder functions $I \longrightarrow A$ by uniform entailment:

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The assignment $I \longmapsto (A^I, \vdash_I)$ *defines the arrow tripos* $T_{\mathcal{A}} \colon \mathbf{Set}^{\mathrm{op}} \longrightarrow \mathbf{Preord}$ *of* $\mathcal{A}$*, with substitution by precomposition and generic predicate* $\mathrm{id}_A \in T_{\mathcal{A}}(A)$.

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Writing $\mathrm{AT}(\mathcal{A}) = \mathbf{Set}[P_{\mathcal{A}}]$, we recover localic and realizability toposes:

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Theorem (Miquel)

Every tripos T *is isomorphic to* $T_{\mathcal{A}}$ *for some (in fact, implicative) arrow algebra* $\mathcal{A}$*.*

Morphisms and nuclei

Morphisms of arrow algebras

Definition

A cartesian morphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a monotone map $f : (A, \preccurlyeq) \longrightarrow (B, \preccurlyeq)$ such that:

1. $f(S_A) \subseteq S_B$;
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A geometric morphism is a left adjoint in $\mathbf{ArrAlg}_{\mathsf{cart}}$, i.e. a cartesian morphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ such that there exists some $h : \mathcal{B} \longrightarrow \mathcal{A}$ satisfying

1. $\mathrm{id}_{\mathcal{A}} \vdash_A h \circ f$;
2. $f \circ h \vdash_{\mathcal{B}} \mathrm{id}_{\mathcal{B}}$.

Morphisms and triposes

A morphism $f\colon \mathcal{A} \longrightarrow \mathcal{B}$ induces a cartesian transformation of triposes $T_{\mathcal{A}} \longrightarrow T_{\mathcal{B}}$, i.e. a pseudonatural transformation whose components preserve finite meets:

$$f \circ -\colon T_{\mathcal{A}}(I) \longrightarrow T_{\mathcal{B}}(I) \qquad \phi \longmapsto f \circ \phi$$

Triposes and cartesian transformations with their order also form a preorder-enriched category $\mathbf{Tripos}_{\mathsf{cart}}$. A geometric transformation is a left adjoint in $\mathbf{Tripos}_{\mathsf{cart}}$.

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Proposition

The assignment $f \longmapsto f \circ -$ gives equivalences of preorders:

$$\mathbf{ArrAlg}_{\mathrm{cart}}(\mathcal{A}, \mathcal{B}) \simeq \mathbf{Tripos}_{\mathrm{cart}}(T_{\mathcal{A}}, T_{\mathcal{B}}) \qquad \mathbf{ArrAlg}_{\mathrm{geom}}(\mathcal{A}, \mathcal{B}) \simeq \mathbf{Tripos}_{\mathrm{geom}}(T_{\mathcal{A}}, T_{\mathcal{B}})$$

and hence of preorder-enriched categories:

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Morphisms between frames

A geometric morphism of toposes $\mathcal{F} \longrightarrow \mathcal{E}$ is a left adjoint functor $\mathcal{E} \longrightarrow \mathcal{F}$ which preserves finite limits. Geometric transformations of triposes always induce geometric morphisms between the associated toposes:

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Since every geometric morphism between localic toposes arises from a geometric transformation of the underlying triposes, we have equivalences:

$$\mathbf{Frm}(X, Y) = \mathbf{ArrAlg}_{\mathrm{geom}}(X, Y) \simeq \mathbf{Tripos}_{\mathrm{geom}}(T_X, T_Y) \simeq \mathbf{Topos}(\mathrm{Sh}(Y), \mathrm{Sh}(X)).$$

Applicative morphisms between PCAs

Let $(P, \cdot, P^{\#})$ and $(Q, \cdot, Q^{\#})$ be PCAs.

A partial applicative morphism $g\colon P \longrightarrow Q$ is a function $g\colon P \longrightarrow \operatorname{Pow} Q$ such that:

1. images of admissible realizers contain admissible realizers:

$$p \in P^{\#} \quad \Longrightarrow \quad g(p) \cap Q^{\#} \neq \varnothing;$$

2. there exists some $r \in Q^{\#}$ tracking application uniformly:

$$p \cdot p' \downarrow,\ a \in g(p),\ b \in g(p') \implies r \cdot a \cdot b \downarrow \ \wedge\ r \cdot a \cdot b \in g(p \cdot p').$$

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Such a morphism is computationally dense if there exists some $m \in B^{\#}$ tracking admissible applications on elements in the image of g, i.e. with the property that for every $s \in B^{\#}$ there is some $r \in A^{\#}$ such that for every $a \in A$:

$$s \cdot g(a) \downarrow \quad \Longrightarrow \quad m \cdot g(r \cdot a) \downarrow\ \wedge\ m \cdot g(r \cdot a) = s \cdot g(a)$$

From PCAs to arrow algebras

A partial applicative morphism $g\colon P \longrightarrow Q$ extends to a cartesian morphism of arrow algebras $\widetilde{g}\colon \mathrm{Pow}\, P \longrightarrow \mathrm{Pow}\, Q$ by setting:

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Proposition

The assignment $g \longmapsto \widetilde{g}$ gives an equivalence of preorders:

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$$\begin{aligned}\mathbf{PCA}_{\mathrm{cd}}(P, Q) &\simeq \mathbf{ArrAlg}_{\mathrm{geom}}(\mathrm{Pow}\, P, \mathrm{Pow}\, Q) \\ &\simeq \mathbf{Tripos}_{\mathrm{geom}}(T_{\mathrm{Pow}\, P}, T_{\mathrm{Pow}\, Q}) \simeq \mathbf{Topos}(\mathrm{RT}(Q), \mathrm{RT}(P))\end{aligned}$$

Subtriposes and subtoposes

Another case in which the map

$$\mathbf{Tripos}_{\mathrm{geom}}(T, T') \longrightarrow \mathbf{Topos}(\mathbf{Set}[T'], \mathbf{Set}[T])$$

restricts to an equivalence is that of *embeddings*:

- a geometric transformation $T \longrightarrow T'$ is an embedding if its right adjoint $T' \longrightarrow T$ is pointwise order-reflecting, in which case T' is a subtripos of T and we write $T' \hookrightarrow T$;
- a geometric morphism $\mathcal{F} \longrightarrow \mathcal{E}$ is an embedding if its right adjoint $\mathcal{F} \longrightarrow \mathcal{E}$ is fully faithful, in which case $\mathcal{F}$ is a subtopos of $\mathcal{E}$ and we write $\mathcal{F} \hookrightarrow \mathcal{E}$.

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We can think of a subtopos $\mathcal{F} \hookrightarrow \mathcal{E}$ as a mathematical universe obtained by imposing additional logical axioms to $\mathcal{E}$. Subtoposes of $\mathcal{E}$ are described internally by local operators on the subobject classifier Ω of $\mathcal{E}$. In the language of arrow algebras, these correspond to nuclei on $\mathcal{A}$.

Nuclei

Definition

A nucleus on an arrow algebra $\mathcal{A}$ is a cartesian morphism $j \colon \mathcal{A} \longrightarrow \mathcal{A}$ such that:

1. $\mathrm{id}_A \vdash_A j$;
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Such a nucleus determines a new arrow algebra $\mathcal{A}_j = (A, \preccurlyeq, \to_j, S_j)$:

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Proposition

The identity map is a cartesian morphism $\mathcal{A} \to \mathcal{A}_j$ inducing a geometric embedding $T_{\mathcal{A}} \longrightarrow T_{\mathcal{A}_j}$. Moreover, every subtripos of $T_{\mathcal{A}}$ arises in this way from an essentially unique nucleus, so that there are equivalences of preorders:

$$\mathrm{Nucl}(\mathcal{A})^{\mathrm{op}} \simeq \mathrm{SubTrip}(T_{\mathcal{A}}) \simeq \mathrm{SubTop}(\mathrm{AT}(\mathcal{A}))$$

Nuclei

Many nuclei of interest have a 'logical' flavor. For instance, for an arrow algebra $\mathcal{A}$ and $u \in A$, the following are nuclei:

- $o_u(x) = u \to x$ gives the open subtopos $\mathsf{AT}(\mathcal{A}_u) \hookrightarrow \mathsf{AT}(\mathcal{A})$, forcing u to be *true*;
- $c_u(x) = x \vee u$ gives the closed subtopos $\mathsf{AT}(\mathcal{A}_{c_u}) \hookrightarrow \mathsf{AT}(\mathcal{A})$, forcing u to be *false*;
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This is the second key advantage of arrow algebras over (quasi-)implicative algebras: $\mathcal{A}_j$ need not be quasi-implicative even if $\mathcal{A}$ is. For an example, take $\operatorname{Pow}\mathcal{K}_1$ and the nucleus $j_\emptyset(x) = (x \to \emptyset) \to \emptyset$.

Modified realizability

Actual and potential realizers

Modified realizability is a variant of Kleene's realizability which, in its untyped collapse, distinguishes two kinds of realizers:

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α : a set of actual realizers, possibly empty,

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Grayson's modified realizability tripos, whose induced topos **Mod** models this interpretation, has set of truth values

$$\Sigma_{\mathbf{Mod}} = \{(\alpha, \beta) \in (\mathsf{Pow}\,\mathbb{N})^2 \mid \alpha \subseteq \beta,\ 0 \in \beta\},$$

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Entailment uses a single realizer which transforms both actual and potential realizers, uniformly across the context.

The Sierpiński construction

We shall now cast this construction in the language of arrow algebras.

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First, for an arrow algebra $\mathcal{A}$, we define its Sierpiński construction as the arrow algebra $\mathsf{S}\mathcal{A}$ on pairs with pointwise evidential order:

$$\mathsf{S}A = \{(x_0, x_1) \in A^2 \mid x_0 \preccurlyeq x_1\},$$

$$(x_0, x_1) \to (y_0, y_1) = \big((x_0 \to y_0) \curlywedge (x_1 \to y_1),\ x_1 \to y_1\big),$$

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For $\mathcal{A} = \mathrm{Pow}\,\mathcal{K}_1$, the coordinates represent actual and potential realizers, but we still miss designated potential realizers. The topos $\mathbf{Eff}_{\to\cdot} = \mathrm{AT}(\mathrm{S}\mathcal{A})$ is the effective topos built 'over the Sierpiński space'.

The modification of an arrow algebra

Proposition

$\mathsf{AT}(\mathcal{A})$ *is the open subtopos of* $\mathsf{AT}(\mathsf{S}\mathcal{A})$ *induced by the nucleus* $o(x) = (\bot, \top) \to x$.

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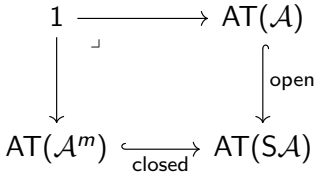
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$\mathsf{AT}(\mathcal{A}^m)$ *is the closed complement of* $\mathsf{AT}(\mathcal{A})$ *among subtoposes of* $\mathsf{AT}(\mathsf{S}\mathcal{A})$.



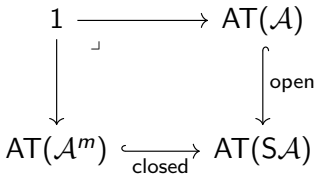
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Eff and **Mod**

For $\mathcal{A} = \mathrm{Pow}\,\mathcal{K}_1$ we have that $\mathsf{AT}(\mathcal{A}^m) \simeq$ **Mod**, and we recover a result originally due to van Oosten: **Mod** is the closed complement of **Eff** in $\mathbf{Eff}_{\to}$.

Functoriality of modified realizability

We can also extend a result originally due to Johnstone in the case of PCAs.

Proposition

Both $\mathcal{A} \mapsto \mathsf{S}\mathcal{A}$ *and* $\mathcal{A} \mapsto \mathcal{A}^m$ *are pseudofunctorial on* $\mathbf{ArrAlg}_{\mathrm{cart}}$ *and* $\mathbf{ArrAlg}_{\mathrm{geom}}$.

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Every geometric morphism of arrow algebras $f : \mathcal{A} \to \mathcal{B}$ therefore induces geometric morphisms and pullback squares of toposes:

$$\begin{array}{ccc} \mathsf{AT}(\mathcal{B}) & \longrightarrow & \mathsf{AT}(\mathcal{A}) \\ \downarrow & & \downarrow \\ \mathsf{AT}(\mathsf{S}\mathcal{B}) & \longrightarrow & \mathsf{AT}(\mathsf{S}\mathcal{A}) \end{array} \qquad \begin{array}{ccc} \mathsf{AT}(\mathcal{B}^m) & \longrightarrow & \mathsf{AT}(\mathcal{A}^m) \\ \downarrow & & \downarrow \\ \mathsf{AT}(\mathsf{S}\mathcal{B}) & \longrightarrow & \mathsf{AT}(\mathsf{S}\mathcal{A}) \end{array}$$

Further directions: extensional modified realizability

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The poset $\mathsf{PER}\,\mathcal{K}_1$ of partial equivalence relations on $\mathcal{K}_1$ also carries the structure of an arrow algebra, whose modification gives an extensional variant to **Mod**:

$$\mathcal{E}_{\mathrm{ext}} = \mathsf{AT}((\mathsf{PER}\,\mathcal{K}_1)^m)$$

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Thank you!

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