Conceptual completeness for geometric logic via ultraconvergence spaces

Umberto Tarantino
jww Sam van Gool, Jérémie Marquès, and Quentin Aristote

IRIF, Université Paris Cité

Making Categorical Model Theory Accessible Workshop
5th June 2026

Introduction and motivation

Barr's *Relational algebras*

In 1970, Barr discusses how to extend a functor $T\colon \mathbf{Set} \to \mathbf{Set}$ to relations by seeing a relation $R \subseteq A \times B$ as the jointly-monic span of its projections $\langle q, p\rangle : R \rightarrowtail A \times B$,

$$\underline{T}\left(A \xleftarrow{q} R \xrightarrow{p} B \right) \triangleq TA \xrightarrow{Tq^{\diamond}} TR \xrightarrow{Tp_{\diamond}} TB$$

where, for a function $f\colon X \to Y$:

- $f_{\diamond} \subseteq X \times Y$ is the *graph* relation, $\{ (x, y) \mid f(x) = y \}$;
- $f^{\diamond} \subseteq Y \times X$ is the *cograph* relation, $\{ (y, x) \mid y = f(x) \}$,

Barr's *Relational algebras*

In 1970, Barr discusses how to extend a functor $T\colon \mathbf{Set} \to \mathbf{Set}$ to relations by seeing a relation $R \subseteq A \times B$ as the jointly-monic span of its projections $\langle q, p\rangle : R \rightarrowtail A \times B$,

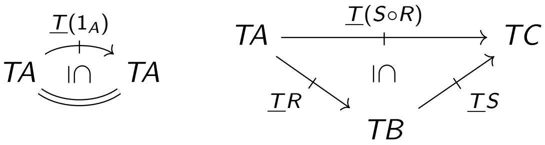
$$\underline{T}\Big(A \xleftarrow{\ q\ } R \xrightarrow{\ p\ } B \Big) \triangleq TA \xrightarrow{Tq^{\diamond}} TR \xrightarrow{Tp_{\diamond}} TB$$

where, for a function $f\colon X \to Y$:

- $f_{\diamond} \subseteq X \times Y$ is the *graph* relation, $\{\, (x, y) \mid f(x) = y \,\}$;
- $f^{\diamond} \subseteq Y \times X$ is the *cograph* relation, $\{\, (y, x) \mid y = f(x) \,\}$,

Proposition (if you believe in the Axiom of Choice)

The extension $\underline{T}\colon \mathbf{Rel} \to \mathbf{Rel}$ *is an oplax functor:*



Barr's *Relational algebras*

In 1970, Barr discusses how to extend a functor $T\colon \mathbf{Set} \to \mathbf{Set}$ to relations by seeing a relation $R \subseteq A \times B$ as the jointly-monic span of its projections $\langle q, p\rangle : R \rightarrowtail A \times B$,

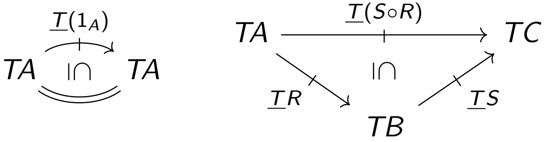
$$\underline{T}\Big(A \xleftarrow{\;q\;} R \xrightarrow{\;p\;} B \Big) \triangleq TA \xrightarrow{Tq^{\diamond}} TR \xrightarrow{Tp_{\diamond}} TB$$

where, for a function $f\colon X \to Y$:

- $f_{\diamond} \subseteq X \times Y$ is the *graph* relation, $\{\, (x, y) \mid f(x) = y \,\}$;
- $f^{\diamond} \subseteq Y \times X$ is the *cograph* relation, $\{\, (y, x) \mid y = f(x) \,\}$,

Proposition (if you don't believe in the Axiom of Choice)

*If T preserves surjections, the extension $\underline{T}\colon$ **Rel** $\to$ **Rel** is an oplax functor:*



Barr's *Relational algebras*

Extensions of natural transformations $\sigma \colon T \Rightarrow T'$ can be treated similarly.

Proposition

The extension $\underline{\sigma} \colon \underline{T} \Rightarrow \underline{T}'$ defined by $\underline{\sigma}_A = (\sigma_A)_\diamond$ is an oplax natural transformation.

$$\begin{array}{ccc} TA & \xrightarrow{\sigma_A} & T'A \\ {\scriptstyle \underline{T}R} \big\downarrow & \subseteq & \big\downarrow {\scriptstyle \underline{T}'R} \\ TB & \xrightarrow[\sigma_B]{} & T'B \end{array}$$

Barr's *Relational algebras*

Extensions of natural transformations $\sigma \colon T \Rightarrow T'$ can be treated similarly.

Proposition

The extension $\underline{\sigma} \colon \underline{T} \Rightarrow \underline{T}'$ defined by $\underline{\sigma}_A = (\sigma_A)_\diamond$ is an oplax natural transformation.

$$\begin{array}{ccc} TA & \xrightarrow{\sigma_A} & T'A \\ {\scriptstyle \underline{T}R}\downarrow & \subseteq & \downarrow{\scriptstyle \underline{T}'R} \\ TB & \xrightarrow[\sigma_B]{} & T'B \end{array}$$

Combining these results, we obtain an extension result for **Set**-monads.

Theorem

*Every **Set**-monad $\langle T, \eta^T, \mu^T \rangle$ extends to a right skew monad $\langle \underline{T}, \eta^{\underline{T}}, \mu^{\underline{T}} \rangle$ on **Rel**, a monad-like structure defined by oplax functors and oplax natural transformations.*

Barr's *Relational algebras*

Barr's main motivation to study these extensions was to recover topological spaces as relational algebras for the ultrafilter monad $\beta\colon \mathbf{Set} \to \mathbf{Set}$.

Theorem

The category **Top** *is isomorphic to the category* $\mathbf{LaxAlg}_{r,co}(\underline{\beta})$ *of lax algebras and representable colax morphisms for the skew monad* $\underline{\beta}\colon \mathbf{Rel} \to \mathbf{Rel}$.

Barr's *Relational algebras*

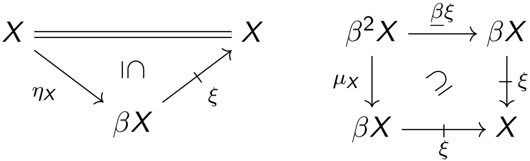
Barr's main motivation to study these extensions was to recover topological spaces as relational algebras for the ultrafilter monad $\beta \colon \mathbf{Set} \to \mathbf{Set}$.

Theorem

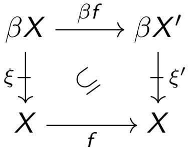
The category **Top** *is isomorphic to the category* $\mathbf{LaxAlg}_{r,co}(\underline{\beta})$ *of lax algebras and representable colax morphisms for the skew monad* $\underline{\beta} \colon \mathbf{Rel} \to \mathbf{Rel}$.

Concretely, this means that:

- a topology on a set X can be equivalently described by a relation $\xi \subseteq \beta X \times X$ such that



- a map $f \colon X \to X'$ is continuous if and only if



Barr's *Relational algebras*

Convergence of ultrafilters

We can think of $\xi \subseteq \beta X \times X$ as specifying a notion of convergence:

- $\nu \; \xi \; x \iff \forall U \subseteq X \text{ open}, \text{ if } x \in U \text{ then } U \in \nu;$
- $U \subseteq X \text{ open} \iff \forall \nu \in \beta X, x \in U, \text{ if } \nu \; \xi \; x \text{ then } U \in \nu.$

Barr's *Relational algebras*

Convergence of ultrafilters

We can think of $\xi \subseteq \beta X \times X$ as specifying a notion of convergence:

- $\nu \; \xi \; x \iff \forall U \subseteq X$ open, if $x \in U$ then $U \in \nu$;
- $U \subseteq X$ open $\iff \forall \nu \in \beta X, x \in U,$ if $\nu \; \xi \; x$ then $U \in \nu$.

For a function $f : I \to X$ and $\nu \in \beta I$, we write $x \prec_{i:\nu} f_i$ in case $\beta f(\nu) \; \xi \; x$ holds:

$$x \prec_{i:\nu} f_i \;\equiv\; \text{“}x \text{ is a limit of } f \text{ along } \nu\text{”}$$

Barr's *Relational algebras*

Convergence of ultrafilters

We can think of $\xi \subseteq \beta X \times X$ as specifying a notion of convergence:

- $\nu \; \xi \; x \iff \forall U \subseteq X \text{ open}, \text{ if } x \in U \text{ then } U \in \nu$;
- $U \subseteq X \text{ open} \iff \forall \nu \in \beta X, x \in U, \text{ if } \nu \; \xi \; x \text{ then } U \in \nu$.

For a function $f \colon I \to X$ and $\nu \in \beta I$, we write $x \prec_{i:\nu} f_i$ in case $\beta f(\nu) \; \xi \; x$ holds:

$$x \prec_{i:\nu} f_i \;\equiv\; \text{“}x \text{ is a limit of } f \text{ along } \nu\text{”}$$

Restricting to *functional* convergence relations, we recover:

- topologically, compact Hausdorff spaces;
- algebraically, (ordinary) β-algebras,

i.e. Manes' theorem $\mathbf{CompHaus} \cong \mathbf{Alg}(\beta)$.

$$\begin{array}{ccc} \mathbf{Top} & \xleftarrow{\;\cong\;}\!\!\rightarrow & \mathbf{LaxAlg}_{r,co}(\underline{\beta}) \\ \uparrow & & \uparrow \\ \mathbf{CompHaus} & \xleftarrow{\;\cong\;}\!\!\rightarrow & \mathbf{Alg}(\beta) \end{array}$$

Now: one dimension higher!

Ultracategories

In 1987, Makkai introduced ultracategories to prove a reconstruction theorem for (coherent) first-order logic.

Ultracategories

In 1987, Makkai introduced ultracategories to prove a reconstruction theorem for (coherent) first-order logic.

- In the propositional case, such a reconstruction theorem is given by Stone duality. For a (classical) theory $\mathbb{T}$, the set $\mathrm{Mod}(\mathbb{T})$ admits a natural Stone topology such that $\mathbf{Stone}(\mathrm{Mod}(\mathbb{T}), \mathbf{2})$ is the (Boolean) Lindenbaum-Tarski algebra of $\mathbb{T}$.

Ultracategories

In 1987, Makkai introduced ultracategories to prove a reconstruction theorem for (coherent) first-order logic.

- In the propositional case, such a reconstruction theorem is given by Stone duality. For a (classical) theory $\mathbb{T}$, the set $\mathrm{Mod}(\mathbb{T})$ admits a natural Stone topology such that $\mathbf{Stone}(\mathrm{Mod}(\mathbb{T}), \mathbf{2})$ is the (Boolean) Lindenbaum-Tarski algebra of $\mathbb{T}$.

For first-order logic, the role of Lindenbaum-Tarski algebras is played by classifying toposes: theories with equivalent classifying toposes have essentially the same models.

Ultracategories

In 1987, Makkai introduced ultracategories to prove a reconstruction theorem for (coherent) first-order logic.

- In the propositional case, such a reconstruction theorem is given by Stone duality. For a (classical) theory $\mathbb{T}$, the set $\mathrm{Mod}(\mathbb{T})$ admits a natural Stone topology such that $\mathbf{Stone}(\mathrm{Mod}(\mathbb{T}), \mathbf{2})$ is the (Boolean) Lindenbaum-Tarski algebra of $\mathbb{T}$.

For first-order logic, the role of Lindenbaum-Tarski algebras is played by classifying toposes: theories with equivalent classifying toposes have essentially the same models.

Theorem (Makkai; Lurie)

For a coherent theory $\mathbb{T}$, the category $\mathbf{Mod}(\mathbb{T})$ is an ultracategory, and $\mathbf{UltCat}(\mathbf{Mod}(\mathbb{T}), \mathbf{Set})$ is the classifying topos of $\mathbb{T}$.

Ultracategories

At its core, an ultracategory is a category C equipped with abstract ultraproducts, i.e. a functorial choice of a C-object $\prod_{i:\nu} c_i$ for each ultrafamily in C, the datum of

- a set I,
- an I-indexed family of C-objects $(c_i)_{i \in I}$, and
- an ultrafilter $\nu \in \beta I$,

behaving as ultraproducts of sets.

Ultracategories

At its core, an ultracategory is a category C equipped with abstract ultraproducts, i.e. a functorial choice of a C-object $\prod_{i:\nu} c_i$ for each ultrafamily in C, the datum of

- a set I,
- an I-indexed family of C-objects $(c_i)_{i \in I}$, and
- an ultrafilter $\nu \in \beta I$,

behaving as ultraproducts of sets.

With appropriate morphisms, the category $\mathbb{\beta} C$ of ultrafamilies in C gives rise to a pseudomonad $\mathbb{\beta} \colon \mathbf{CAT} \to \mathbf{CAT}$, dubbed the ultracompletion pseudomonad by Rosolini.

Ultracategories

At its core, an ultracategory is a category C equipped with abstract ultraproducts, i.e. a functorial choice of a C-object $\prod_{i:\nu} c_i$ for each ultrafamily in C, the datum of

- a set I,
- an I-indexed family of C-objects $(c_i)_{i \in I}$, and
- an ultrafilter $\nu \in \beta I$,

behaving as ultraproducts of sets.

With appropriate morphisms, the category βC of ultrafamilies in C gives rise to a pseudomonad $\beta\colon \mathbf{CAT} \to \mathbf{CAT}$, dubbed the ultracompletion pseudomonad by Rosolini.

Definition (Garner, Rosolini)

UltCat is the 2-category of pseudo-β-algebras, colax morphisms, and algebra 2-cells.

Ultracategories

At its core, an ultracategory is a category C equipped with abstract ultraproducts, i.e. a functorial choice of a C-object $\prod_{i:\nu} c_i$ for each ultrafamily in C, the datum of

- a set I,
- an I-indexed family of C-objects $(c_i)_{i\in I}$, and
- an ultrafilter $\nu \in \beta I$,

behaving as ultraproducts of sets.

With appropriate morphisms, the category βC of ultrafamilies in C gives rise to a pseudomonad $\beta\colon \mathbf{CAT} \to \mathbf{CAT}$, dubbed the ultracompletion pseudomonad by Rosolini.

Definition (Garner, Rosolini)

UltCat is the 2-category of pseudo-β-algebras, colax morphisms, and algebra 2-cells.

Ultracategories categorify compact Hausdorff spaces

CompHaus $\hookrightarrow$ **UltCat** as the small and discrete ultracategories.

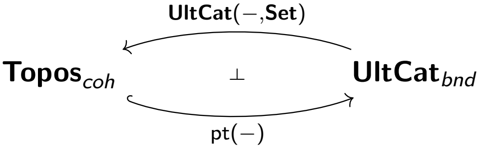
Ultracategories and coherent toposes

Identifying coherent theories with coherent toposes, and restricting to the subcategory $\mathbf{UltCat}_{bnd}$ of ultracategories C such that $\mathbf{UltCat}(C, \mathbf{Set})$ is a topos, we have:

$$\mathbf{Topos}_{coh} \underset{\mathrm{pt}(-)}{\overset{\mathbf{UltCat}(-,\mathbf{Set})}{\underset{\hookrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{UltCat}_{bnd}$$

Ultracategories and coherent toposes

Identifying coherent theories with coherent toposes, and restricting to the subcategory $\mathbf{UltCat}_{bnd}$ of ultracategories C such that $\mathbf{UltCat}(C, \mathbf{Set})$ is a topos, we have:



This result crucially rests on Łoś's theorem: for a coherent theory, ultraproducts of models are themselves models.

Ultracategories and coherent toposes

Identifying coherent theories with coherent toposes, and restricting to the subcategory $\mathbf{UltCat}_{bnd}$ of ultracategories C such that $\mathbf{UltCat}(C, \mathbf{Set})$ is a topos, we have:

$$\mathbf{Topos}_{coh} \underset{\mathrm{pt}(-)}{\overset{\mathbf{UltCat}(-,\mathbf{Set})}{\rightleftarrows}} \mathbf{UltCat}_{bnd} \quad (\perp)$$

This result crucially rests on Łoś's theorem: for a coherent theory, ultraproducts of models are themselves models. In other words, the ultraproduct functors

$$\mathbf{Set}^I \xrightarrow{\prod_{i:\nu}(-)} \mathbf{Set}$$

are coherent, i.e. they preserve finite limits, regular epis, and finite unions of subobjects.

What about geometric logic?

We will now consider geometric logic: a theory is geometric if its axioms are of the form $\forall \vec{x}(\varphi(\vec{x}) \to \psi(\vec{x}))$ where φ, ψ are built only using finitary $\wedge$, infinitary $\vee$, and $\exists$.

For a geometric theory, Łoś's theorem fails: the ultraproduct functors are not geometric, as they don't preserve arbitrary unions of subobjects.

What about geometric logic?

We will now consider geometric logic: a theory is geometric if its axioms are of the form $\forall \vec{x}(\varphi(\vec{x}) \to \psi(\vec{x}))$ where φ, ψ are built only using finitary $\wedge$, infinitary $\vee$, and $\exists$.

For a geometric theory, Łoś's theorem fails: the ultraproduct functors are not geometric, as they don't preserve arbitrary unions of subobjects.

Question

Can we extend the previous to a reconstruction theorem for geometric logic?

What about geometric logic?

We will now consider geometric logic: a theory is geometric if its axioms are of the form $\forall\vec{x}(\varphi(\vec{x}) \rightarrow \psi(\vec{x}))$ where φ, ψ are built only using finitary $\wedge$, infinitary $\vee$, and $\exists$. For a geometric theory, Łoś's theorem fails: the ultraproduct functors are not geometric, as they don't preserve arbitrary unions of subobjects.

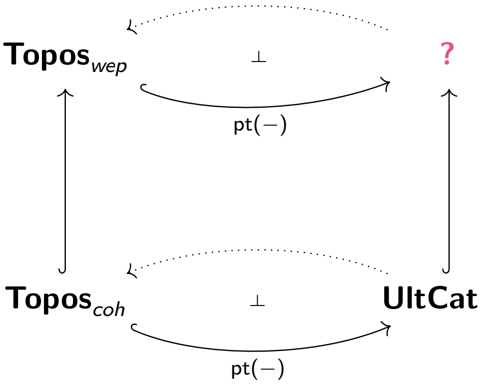
Question

Can we extend the previous to a reconstruction theorem for geometric logic?

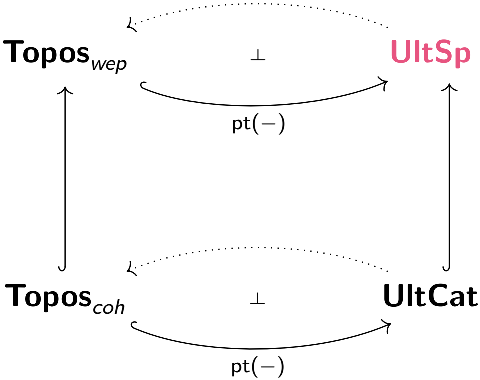
A necessary restriction

Having such a result, for a geometric theory $\mathbb{T}$, entails its completeness with respect to its **Set**-models. Categorically, this corresponds to restricting to toposes with enough points, a condition analogous to *spatiality* for locales.

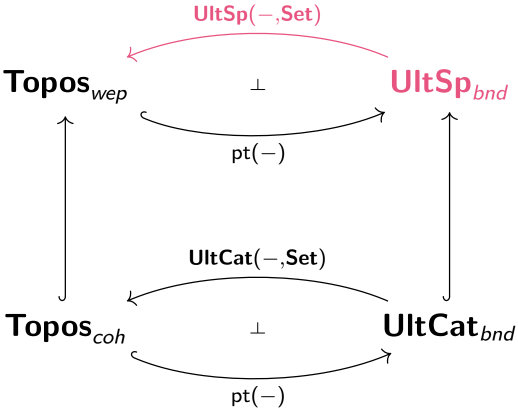
What about geometric logic?



What about geometric logic?



What about geometric logic?



Reconstruction via ultraconvergence spaces

Ultraconvergence spaces

The key intuition to address this question comes from Barr's theorem: we can generalize ultracategories by replacing a β-algebra functor with a profunctor, analogue of a relation between categories, and categorifying the notion of convergence.

Ultraconvergence spaces

The key intuition to address this question comes from Barr's theorem: we can generalize ultracategories by replacing a β-algebra functor with a profunctor, analogue of a relation between categories, and categorifying the notion of convergence.

Definition

An ultraconvergence space consists of a discrete category X together with a profunctor $\Xi \colon \beta X \nrightarrow X$,

Ultraconvergence spaces

The key intuition to address this question comes from Barr's theorem: we can generalize ultracategories by replacing a β-algebra functor with a profunctor, analogue of a relation between categories, and categorifying the notion of convergence.

Definition

An ultraconvergence space consists of a discrete category X together with a profunctor $\Xi \colon \beta X \nrightarrow X$,

Ultraconvergence spaces

The key intuition to address this question comes from Barr's theorem: we can generalize ultracategories by replacing a β-algebra functor with a profunctor, analogue of a relation between categories, and categorifying the notion of convergence.

Definition

An ultraconvergence space consists of a discrete category X together with a profunctor $\Xi\colon X \times \beta X \longrightarrow \mathbf{Set}$,

Ultraconvergence spaces

The key intuition to address this question comes from Barr's theorem: we can generalize ultracategories by replacing a β-algebra functor with a profunctor, analogue of a relation between categories, and categorifying the notion of convergence.

Definition

An ultraconvergence space consists of a discrete category X together with a profunctor $\Xi\colon X \times \beta X \longrightarrow \mathbf{Set}$, where elements of $\Xi(x, (I, y, \nu))$ are dubbed *ultra-arrows* and denoted by $r\colon x \prec_{i:\nu} y_i$.

Ultraconvergence spaces

The key intuition to address this question comes from Barr's theorem: we can generalize ultracategories by replacing a β-algebra functor with a profunctor, analogue of a relation between categories, and categorifying the notion of convergence.

Definition

An ultraconvergence space consists of a discrete category X together with a profunctor $\Xi\colon X \times \beta X \longrightarrow \mathbf{Set}$, where elements of $\Xi(x, (I, y, \nu))$ are dubbed *ultra-arrows* and denoted by $r\colon x \prec_{i:\nu} y_i$. Moreover, X is equipped with:

- for every $x \in X$, an *identity* ultra-arrow $\mathrm{id}_x\colon x \prec_{*:1} x$;
- for every ultra-arrow $r\colon x \prec_{i:\nu} y_i$ and every ultrafamily of ultra-arrows $(s_i\colon y_i \prec_{j:\zeta_i} z_{i,j})_{i:\nu}$, a *composite* ultra-arrow $(s_i)_{i:\nu} \cdot r\colon x \prec_{(i,j):\sum_{i:\nu}\zeta_i} z_{i,j}$,

satisfying equational axioms which express unitality and associativity of composition:

1. $(r)_{*:1} \cdot \mathrm{id}_x = r$,
2. ...

Continuous maps

Similarly, we can extend the notion of continuity to this **Set**-valued convergence relation, which now becomes *structure* rather than *property*.

A continuous map of ultraconvergence spaces is a functor $f\colon X \longrightarrow X'$ together with a mapping $(r\colon x \prec_{i:\nu} y_i) \mapsto (f(r)\colon f(x) \prec_{i:\nu} f(y_i))$ on ultra-arrows, also satisfying equational axioms:

1. $f(\mathrm{id}_x) = \mathrm{id}_{f(x)}$,
2. ...

Continuous maps

Similarly, we can extend the notion of continuity to this **Set**-valued convergence relation, which now becomes *structure* rather than *property*.

A continuous map of ultraconvergence spaces is a functor $f\colon X \longrightarrow X'$ together with a mapping $(r\colon x \prec_{i:\nu} y_i) \mapsto (f(r)\colon f(x) \prec_{i:\nu} f(y_i))$ on ultra-arrows, also satisfying equational axioms:

1. $f(\mathrm{id}_x) = \mathrm{id}_{f(x)}$,
2. $\ldots$

A transformation of continuous maps $\alpha\colon f \Rightarrow f'$ is a family of ultra-arrows $\alpha_x\colon f(x) \prec_{*:1} f'(x)$ in X' for each $x \in X$, such that for each ultra-arrow $r\colon x \prec_{i:\nu} x_i$:

$$f'(r) \cdot \alpha_x = (\alpha_{x_i})_{i:\nu} \cdot f(r)$$

Continuous maps

Similarly, we can extend the notion of continuity to this **Set**-valued convergence relation, which now becomes *structure* rather than *property*.

A continuous map of ultraconvergence spaces is a functor $f\colon X \longrightarrow X'$ together with a mapping $(r\colon x \prec_{i:\nu} y_i) \mapsto (f(r)\colon f(x) \prec_{i:\nu} f(y_i))$ on ultra-arrows, also satisfying equational axioms:

1. $f(\mathrm{id}_x) = \mathrm{id}_{f(x)}$,
2. ...

A transformation of continuous maps $\alpha\colon f \Rightarrow f'$ is a family of ultra-arrows $\alpha_x\colon f(x) \prec_{*:1} f'(x)$ in X' for each $x \in X$, such that for each ultra-arrow $r\colon x \prec_{i:\nu} x_i$:

$$f'(r) \cdot \alpha_x = (\alpha_{x_i})_{i:\nu} \cdot f(r)$$

UltSp is the 2-category of ultraconvergence spaces, continuous maps, and transformations.

Examples

- Every topological space is a **2**-valued ultraconvergence space with the usual convergence relation.

Examples

- Every topological space is a **2**-valued ultraconvergence space with the usual convergence relation.
- Every ultracategory C, defined by a functor $\Phi\colon \beta C \longrightarrow C$, is an ultraconvergence space by setting ultra-arrows $c \prec_{i:\nu} d_i$ to be C-arrows $c \to \Phi(I, d, \nu)$.

Examples

- Every topological space is a **2**-valued ultraconvergence space with the usual convergence relation.
- Every ultracategory C, defined by a functor $\Phi\colon \beta C \longrightarrow C$, is an ultraconvergence space by setting ultra-arrows $c \prec_{i:\nu} d_i$ to be C-arrows $c \to \Phi(I, d, \nu)$.

$$\begin{array}{ccc}
\mathbf{UltCat} & \hookrightarrow & \mathbf{UltSp} \\
\uparrow & & \uparrow \\
\mathbf{CompHaus} & \hookrightarrow & \mathbf{Top}
\end{array}$$

The main theorem

As promised, the notion of ultraconvergence space allows us to obtain a reconstruction theorem for geometric logic: in topos-theoretical terms, it reads as follows.

Theorem (Saadia; Hamad; van Gool, Marquès, T.)

If $\mathcal{E}$ is a topos with enough points, then $\mathcal{E} \simeq \mathbf{UltSp}(\mathrm{pt}(\mathcal{E}), \mathbf{Set})$.

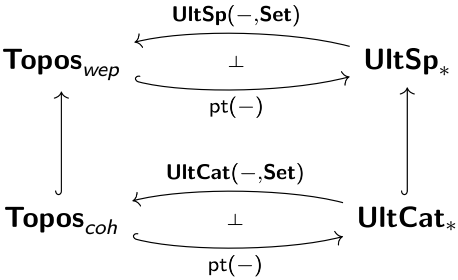
The main theorem

As promised, the notion of ultraconvergence space allows us to obtain a reconstruction theorem for geometric logic: in topos-theoretical terms, it reads as follows.

Theorem (Saadia; Hamad; van Gool, Marquès, T.)

If $\mathcal{E}$ is a topos with enough points, then $\mathcal{E} \simeq \mathbf{UltSp}(\mathrm{pt}(\mathcal{E}), \mathbf{Set})$.

In other words, restricting to the subcategory $\mathbf{UltSp}_*$ of ultraconvergence spaces X such that $\mathbf{UltSp}(X, \mathbf{Set})$ is a topos, we have what we wanted:



Étale spaces

Differently than Saadia's and Hamad's approaches, our proof crucially relies on a Grothendieck correspondence for continuous maps towards **Set**, which yields an extension of local homeomorphisms from topological to ultraconvergence spaces.

Étale spaces

Differently than Saadia's and Hamad's approaches, our proof crucially relies on a Grothendieck correspondence for continuous maps towards **Set**, which yields an extension of local homeomorphisms from topological to ultraconvergence spaces.

Definition

A continuous map of ultraconvergence spaces $\pi\colon E \longrightarrow B$ is étale if:

Étale spaces

Differently than Saadia's and Hamad's approaches, our proof crucially relies on a Grothendieck correspondence for continuous maps towards **Set**, which yields an extension of local homeomorphisms from topological to ultraconvergence spaces.

Definition

A continuous map of ultraconvergence spaces $\pi \colon E \longrightarrow B$ is étale if:

1. for each $b \in B$, the fiber $\pi^{-1}(b)$ is a set;

Étale spaces

Differently than Saadia's and Hamad's approaches, our proof crucially relies on a Grothendieck correspondence for continuous maps towards **Set**, which yields an extension of local homeomorphisms from topological to ultraconvergence spaces.

Definition

A continuous map of ultraconvergence spaces $\pi\colon E \longrightarrow B$ is étale if:

1. for each $b \in B$, the fiber $\pi^{-1}(b)$ is a set;
2. for each $e \in E$ and each ultra-arrow $r\colon \pi(e) \prec_{i:\nu} b_i$ in B, there is a unique *lift* $\bar{r}\colon e \prec_{i:\nu} e_i$ in E such that $\pi(\bar{r}) = r$.

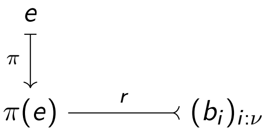
Étale spaces

Differently than Saadia's and Hamad's approaches, our proof crucially relies on a Grothendieck correspondence for continuous maps towards **Set**, which yields an extension of local homeomorphisms from topological to ultraconvergence spaces.

Definition

A continuous map of ultraconvergence spaces $\pi\colon E \longrightarrow B$ is étale if:

1. for each $b \in B$, the fiber $\pi^{-1}(b)$ is a set;
2. for each $e \in E$ and each ultra-arrow $r\colon \pi(e) \prec_{i:\nu} b_i$ in B, there is a unique *lift* $\bar{r}\colon e \prec_{i:\nu} e_i$ in E such that $\pi(\bar{r}) = r$.



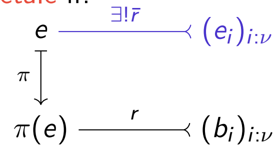
Étale spaces

Differently than Saadia's and Hamad's approaches, our proof crucially relies on a Grothendieck correspondence for continuous maps towards **Set**, which yields an extension of local homeomorphisms from topological to ultraconvergence spaces.

Definition

A continuous map of ultraconvergence spaces $\pi\colon E \longrightarrow B$ is étale if:

1. for each $b \in B$, the fiber $\pi^{-1}(b)$ is a set;
2. for each $e \in E$ and each ultra-arrow $r\colon \pi(e) \prec_{i:\nu} b_i$ in B, there is a unique *lift* $\bar{r}\colon e \prec_{i:\nu} e_i$ in E such that $\pi(\bar{r}) = r$.



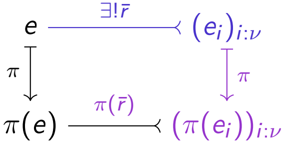
Étale spaces

Differently than Saadia's and Hamad's approaches, our proof crucially relies on a Grothendieck correspondence for continuous maps towards **Set**, which yields an extension of local homeomorphisms from topological to ultraconvergence spaces.

Definition

A continuous map of ultraconvergence spaces $\pi\colon E \longrightarrow B$ is étale if:

1. for each $b \in B$, the fiber $\pi^{-1}(b)$ is a set;
2. for each $e \in E$ and each ultra-arrow $r\colon \pi(e) \prec_{i:\nu} b_i$ in B, there is a unique *lift* $\bar{r}\colon e \prec_{i:\nu} e_i$ in E such that $\pi(\bar{r}) = r$.



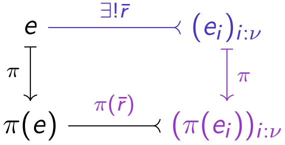
Étale spaces

Differently than Saadia's and Hamad's approaches, our proof crucially relies on a Grothendieck correspondence for continuous maps towards **Set**, which yields an extension of local homeomorphisms from topological to ultraconvergence spaces.

Definition

A continuous map of ultraconvergence spaces $\pi \colon E \longrightarrow B$ is étale if:

1. for each $b \in B$, the fiber $\pi^{-1}(b)$ is a set;
2. for each $e \in E$ and each ultra-arrow $r \colon \pi(e) \prec_{i:\nu} b_i$ in B, there is a unique *lift* $\bar{r} \colon e \prec_{i:\nu} e_i$ in E such that $\pi(\bar{r}) = r$.



Étale maps over B form a category $\mathbf{Et}(B)$, equivalent to $\mathbf{UltSp}(B, \mathbf{Set})$.

The ultraconvergence space of points of a topos

Let $\mathcal{E}$ be a topos with a fixed class of points $X \subseteq \mathrm{pt}(\mathcal{E})$.

The ultraconvergence space of points of a topos

Let $\mathcal{E}$ be a topos with a fixed class of points $X \subseteq \mathrm{pt}(\mathcal{E})$.

- X is an ultraconvergence space by setting ultra-arrows $x \prec_{i:\nu} y_i$ to be natural transformations $x \Rightarrow \prod_{i:\nu} y_i$, where $\prod_{i;\nu} y_i$ is the functor:

$$\mathcal{E} \xrightarrow{\langle y_i \rangle_{i \in I}} \mathbf{Set}^I \xrightarrow{\prod_{i:\nu}(-)} \mathbf{Set}$$

The ultraconvergence space of points of a topos

Let $\mathcal{E}$ be a topos with a fixed class of points $X \subseteq \mathrm{pt}(\mathcal{E})$.

- X is an ultraconvergence space by setting ultra-arrows $x \prec_{i:\nu} y_i$ to be natural transformations $x \Rightarrow \prod_{i:\nu} y_i$, where $\prod_{i;\nu} y_i$ is the functor:

$$\mathcal{E} \xrightarrow{\langle y_i \rangle_{i \in I}} \mathbf{Set}^I \xrightarrow{\prod_{i:\nu}(-)} \mathbf{Set}$$

- For every object $\varphi \in \mathcal{E}$, we can define an étale space $\pi_\varphi \colon [\![\varphi]\!] \longrightarrow X$ where:
  - the fiber of π_φ at $x \in X$ is given by $x(\varphi)$;
  - an ultra-arrow $(x, v) \prec_{i:\nu} (y_i, w_i)$ in $[\![\varphi]\!]$ is given by an ultra-arrow $r \colon x \prec_{i:\nu} y_i$ in X such that $r_\varphi(v) = (w_i)_{i:\nu}$.

This assignment defines the *evaluation functor* $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$.

Reconstruction for geometric logic

Theorem

If X is a separating set of points of $\mathcal{E}$, then $[\![-]\!]\colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is an equivalence.

Reconstruction for geometric logic

Theorem

If X is a separating set of points of $\mathcal{E}$, then $[\![-]\!]\colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is an equivalence.

Although we need X to be small to prove the above result, it follows easily that $\mathcal{E} \simeq \mathbf{Et}(\mathrm{pt}(\mathcal{E}))$. In logical terms, this reads as the following reconstruction result.

Theorem

Let $\mathbb{T}$ be a geometric theory which is complete with respect to its **Set***-models. Then, $\mathrm{Mod}(\mathbb{T})$ is an ultraconvergence space by setting ultra-arrows $M \prec_{i:\nu} N_i$ to be structure morphisms $M \to \prod_{i:\nu} N_i$, and $\mathbf{Et}(\mathrm{Mod}(\mathbb{T}))$ is the classifying topos of $\mathbb{T}$.*

Reconstruction for geometric logic

Theorem

If X is a separating set of points of $\mathcal{E}$, then $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is an equivalence.

Although we need X to be small to prove the above result, it follows easily that $\mathcal{E} \simeq \mathbf{Et}(\mathrm{pt}(\mathcal{E}))$. In logical terms, this reads as the following reconstruction result.

Theorem

*Let $\mathbb{T}$ be a geometric theory which is complete with respect to its **Set**-models. Then, $\mathrm{Mod}(\mathbb{T})$ is an ultraconvergence space by setting ultra-arrows $M \prec_{i:\nu} N_i$ to be structure morphisms $M \to \prod_{i:\nu} N_i$, and $\mathbf{Et}(\mathrm{Mod}(\mathbb{T}))$ is the classifying topos of $\mathbb{T}$.*

The localic/propositional case

In particular, if a localic topos $\mathcal{E}$ has enough points, i.e. $\mathcal{E} \simeq \mathrm{Sh}(\mathcal{O}(X))$ for some topological space X, then $\mathcal{E} \simeq \mathrm{Et}(X)$.

Proof sketch

Our proof proceeds similarly to Makkai's original work, in two main steps.

Proof sketch

Our proof proceeds similarly to Makkai's original work, in two main steps.

1. $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is full on subobjects: every subobject of $\pi_\varphi \colon [\![\varphi]\!] \longrightarrow X$ in $\mathrm{Et}(X)$ is the restriction of π_φ to $[\![\psi]\!] \subseteq [\![\varphi]\!]$ for some subobject $\psi \rightarrowtail \varphi$ in $\mathcal{E}$.

Proof sketch

Our proof proceeds similarly to Makkai's original work, in two main steps.

1. $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is full on subobjects: every subobject of $\pi_\varphi \colon [\![\varphi]\!] \longrightarrow X$ in $\mathrm{Et}(X)$ is the restriction of π_φ to $[\![\psi]\!] \subseteq [\![\varphi]\!]$ for some subobject $\psi \rightarrowtail \varphi$ in $\mathcal{E}$.
2. $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is covering: every étale space $p \colon Y \longrightarrow X$ is covered by an epimorphism $\alpha \colon \pi_\varphi \twoheadrightarrow p$ in $\mathrm{Et}(X)$ for some object $\varphi \in \mathcal{E}$.

Proof sketch

Our proof proceeds similarly to Makkai's original work, in two main steps.

1. $[\![-]\!]\colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is full on subobjects: every subobject of $\pi_\varphi\colon [\![\varphi]\!] \longrightarrow X$ in $\mathrm{Et}(X)$ is the restriction of π_φ to $[\![\psi]\!] \subseteq [\![\varphi]\!]$ for some subobject $\psi \rightarrowtail \varphi$ in $\mathcal{E}$.
2. $[\![-]\!]\colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is covering: every étale space $p\colon Y \longrightarrow X$ is covered by an epimorphism $\alpha\colon \pi_\varphi \twoheadrightarrow p$ in $\mathrm{Et}(X)$ for some object $\varphi \in \mathcal{E}$.

Two points of view

Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $[\![-]\!]$.

Proof sketch

Our proof proceeds similarly to Makkai's original work, in two main steps.

1. $[\![-]\!]\colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is full on subobjects: every subobject of $\pi_\varphi \colon [\![\varphi]\!] \longrightarrow X$ in $\mathrm{Et}(X)$ is the restriction of π_φ to $[\![\psi]\!] \subseteq [\![\varphi]\!]$ for some subobject $\psi \rightarrowtail \varphi$ in $\mathcal{E}$.
2. $[\![-]\!]\colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is covering: every étale space $p\colon Y \longrightarrow X$ is covered by an epimorphism $\alpha\colon \pi_\varphi \twoheadrightarrow p$ in $\mathrm{Et}(X)$ for some object $\varphi \in \mathcal{E}$.

Two points of view

Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $[\![-]\!]$. However, we can also interpret (1) as stating that $[\![-]\!]$ defines a hyperconnected geometric morphism, and (2) as stating that it defines a localic geometric morphism.

Ultraconvergence spaces as profunctorial algebras

Extensions *à la* Barr

As it turns out, the inspiration from Barr's theorem can be pushed even further, to a 2-dimensional analogous extension theorem.

Extensions *à la* Barr

As it turns out, the inspiration from Barr's theorem can be pushed even further, to a 2-dimensional analogous extension theorem. Representing profunctors via *cospans* rather than spans, we can extend a pseudofunctor $\mathbf{T}\colon \mathbf{CAT} \to \mathbf{CAT}$ by setting:

$$\underline{\mathbf{T}}\left(C \xrightarrow{\iota_C} E \xleftarrow{\iota_D} D \right) \triangleq \mathbf{T}C \xrightarrow{(\mathbf{T}\iota_C)_\diamond} \mathbf{T}E \xrightarrow{(\mathbf{T}\iota_D)^\diamond} \mathbf{T}D$$

Here, for a functor $f\colon X \to Y$:

- $f_\diamond\colon X \nrightarrow Y$ is the *graph* profunctor, $f_\diamond(y, x) = Y(y, f(x))$;
- $f^\diamond\colon Y \nrightarrow X$ is the *cograph* profunctor, $f^\diamond(x, y) = Y(f(x), y)$,

which give rise to locally fully-faithful pseudofunctors:

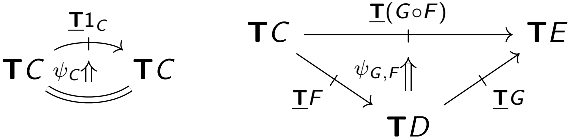
$$(-)_\diamond\colon \mathbf{CAT} \to \mathbf{PROF} \qquad (-)^\diamond\colon \mathbf{CAT}^{\mathrm{op}} \to \mathbf{PROF}^{\mathrm{co}}$$

Extending pseudofunctors

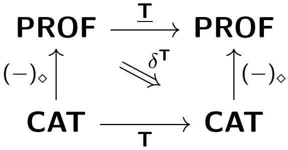
Barr's preservation of surjections here becomes preservation of fully-faithful functors.

Proposition

If $\mathbf{T}$ *preserves fully-faithfulness, the extension* $\underline{\mathbf{T}}\colon \mathbf{PROF} \to \mathbf{PROF}$ *is a lax functor:*



Moreover, $\underline{\mathbf{T}}$ *is equipped with a lax natural transformation* $\delta^{\mathbf{T}}\colon \underline{\mathbf{T}}(-)_\diamond \Rightarrow (\mathbf{T}-)_\diamond$*:*



Extending transformations and modifications

Extensions of pseudonatural transformations $\sigma\colon \mathbf{T} \Rightarrow \mathbf{T}'$ and modifications $\mathfrak{m}\colon \sigma \Rrightarrow \sigma'$ can be treated similarly, simply by carrying them along $(-)_\diamond\colon \mathbf{CAT} \to \mathbf{PROF}$.

Proposition

The extension $\underline{\sigma}\colon \underline{\mathbf{T}} \Rightarrow \underline{\mathbf{T}}'$ defined by $\underline{\sigma}_C = (\sigma_C)_\diamond$ is a lax natural transformation.

$$\begin{array}{ccc} \mathbf{T}C & \xrightarrow{\sigma_C} & \mathbf{T}'C \\ {\scriptstyle \underline{\mathbf{T}}F}\big\downarrow\!\!\!\!+ & \underline{\sigma}_F \swarrow & +\!\!\!\!\big\downarrow{\scriptstyle \underline{\mathbf{T}}'F} \\ \mathbf{T}D & \xrightarrow[\sigma_D]{} & \mathbf{T}'D \end{array}$$

Proposition

The extension $\underline{\mathfrak{m}}\colon \underline{\sigma} \Rrightarrow \underline{\sigma}'$ defined by $\underline{\mathfrak{m}}_C = (\mathfrak{m}_C)_\diamond$ is a modification.

Extending pseudomonads

Combining these results, we obtain an extension result for pseudomonads on **CAT**.

Theorem (Aristote, T.)

Every pseudomonad $\langle \mathbf{T}, \eta^{\mathbf{T}}, \mu^{\mathbf{T}} \rangle$ *on* **CAT** *preserving fully-faithfulness extends to a left skew monad* $\langle \underline{\mathbf{T}}, \eta^{\underline{\mathbf{T}}}, \mu^{\underline{\mathbf{T}}} \rangle$ *on* **PROF**.

Extending pseudomonads

Combining these results, we obtain an extension result for pseudomonads on **CAT**.

Theorem (Aristote, T.)

Every pseudomonad $\langle \mathbf{T}, \eta^{\mathbf{T}}, \mu^{\mathbf{T}} \rangle$ *on* **CAT** *preserving fully-faithfulness extends to a left skew monad* $\langle \underline{\mathbf{T}}, \eta^{\underline{\mathbf{T}}}, \mu^{\underline{\mathbf{T}}} \rangle$ *on* **PROF**.

In particular, mimicking Barr, we can apply it to β.

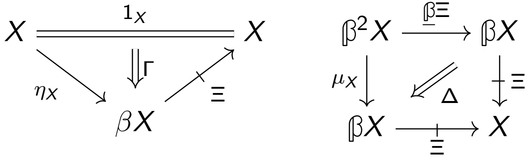
Corollary

The ultracompletion pseudomonad β *extends to a left skew monad* $\underline{\beta}$ *on* **PROF**.

Ultraconvergence spaces as discrete algebras

Fact

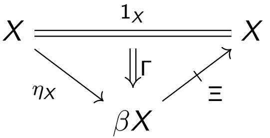
The structure of an ultraconvergence space on a discrete category X coincides with that of a lax $\underline{\mathbb{\beta}}$-algebra on X, i.e. a profunctor $\Xi\colon \mathbb{\beta}X \nrightarrow X$ together with two natural transformations:

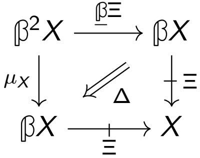


Ultraconvergence spaces as discrete algebras

Fact

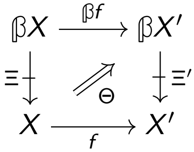
The structure of an ultraconvergence space on a discrete category X coincides with that of a lax $\underline{\beta}$-algebra on X, i.e. a profunctor $\Xi\colon \beta X \nrightarrow X$ together with two natural transformations:





Fact

The structure of a continuous map on a functor $f\colon X \to X'$ coincides with that of a colax morphism of lax $\underline{\beta}$-algebras on f, i.e. a natural transformation:



Now what?

The previous facts yield two problems.

Now what?

The previous facts yield two problems.

First problem

The notions of lax algebras and their colax morphisms are not well-defined for a left skew monad which is not a pseudomonad.

Now what?

The previous facts yield two problems.

First problem

The notions of lax algebras and their colax morphisms are not well-defined for a left skew monad which is not a pseudomonad.

Second problem

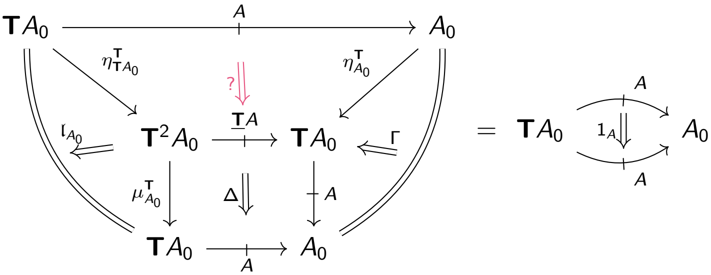
Algebra 2-cells, however defined, trivialize. How do transformations between continuous maps arise?

A coherence problem

Unlike the 1-dimensional case, here we have to consider appropriate coherence conditions which are not evident in this lax setting unless we assume more properties.

A concrete example

For a lax $\underline{\mathbf{T}}$-algebra $A\colon \mathbf{T}A_0 \nrightarrow A_0$, there should be a ‘right unitality’ axiom



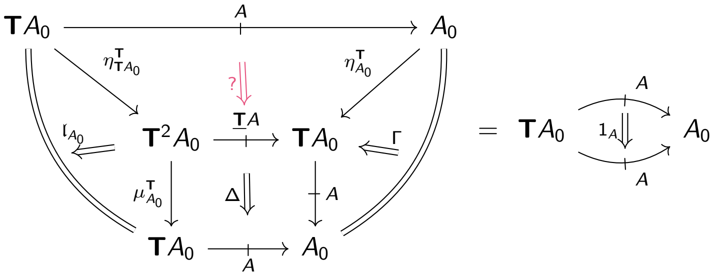
but, being $\eta^{\underline{\mathbf{T}}}$ *lax* natural, its structural 2-cell goes in the wrong way.

A coherence problem

Unlike the 1-dimensional case, here we have to consider appropriate coherence conditions which are not evident in this lax setting unless we assume more properties.

A concrete example

For a lax $\underline{\mathbf{T}}$-algebra $A\colon \mathbf{T}A_0 \nrightarrow A_0$, there should be a 'right unitality' axiom



but, being $\eta^{\underline{\mathbf{T}}}$ *lax* natural, its structural 2-cell goes in the wrong way.

- For ultraconvergence spaces, this should read as $(r)_{*:1} \cdot \mathrm{id}_x = r$ for all $r\colon x \prec_{i:\nu} y_i$.

If we allow for lax algebras...

Definition

The skew extension $\langle \underline{\mathbf{T}}, \eta^{\underline{\mathbf{T}}}, \mu^{\underline{\mathbf{T}}} \rangle$ of a pseudomonad $\langle \mathbf{T}, \eta^{\mathbf{T}}, \mu^{\mathbf{T}} \rangle$ allows for lax algebras if:

1. the unit $\eta^{\underline{\mathbf{T}}}$ and the multiplication $\mu^{\underline{\mathbf{T}}}$ are pseudonatural;
2. the transformation $\delta^{\mathbf{T}} \colon \underline{\mathbf{T}}(-)_\diamond \Rightarrow (\mathbf{T}-)_\diamond$ is pseudonatural;
3. there exists a natural family of natural transformations $\varphi_{G,f_\diamond} \colon \underline{\mathbf{T}}(G \circ f_\diamond) \Rightarrow \underline{\mathbf{T}}G \circ \underline{\mathbf{T}}f_\diamond$, for any functor $f \colon A \to B$ and any profunctor $G \colon B \nrightarrow C$, such that for each pair of functors $f \colon A \to B, f' \colon B \to C$ and each profunctor $G \colon C \nrightarrow D$:

   3.1 the diagram below commutes;

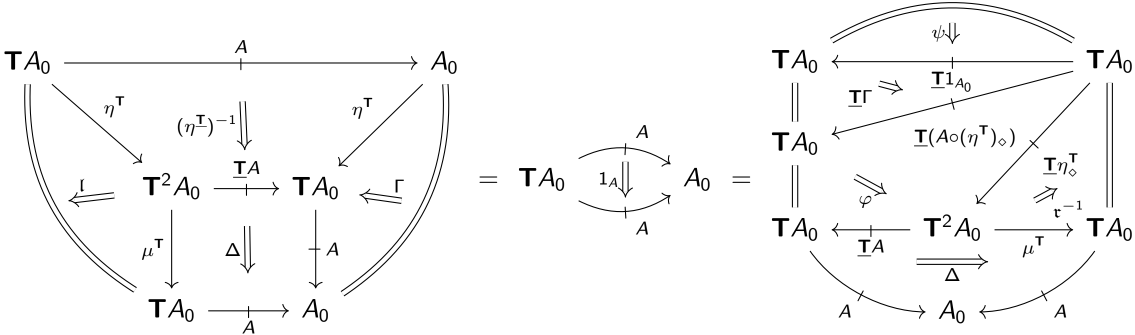
$$\begin{array}{ccccc}
\underline{\mathbf{T}}(G \circ (f'f)_\diamond) & \xrightarrow{\varphi_{G,(f'f)_\diamond}} & \underline{\mathbf{T}}G \circ \underline{\mathbf{T}}(f'f)_\diamond & \xrightarrow{\cong} & \underline{\mathbf{T}}G \circ \underline{\mathbf{T}}(f'_\diamond \circ f_\diamond) \\
{\scriptstyle \cong}\downarrow & & & & \downarrow{\scriptstyle \underline{\mathbf{T}}G * \varphi_{f'_\diamond, f_\diamond}} \\
\underline{\mathbf{T}}(G \circ f'_\diamond \circ f_\diamond) & \xrightarrow[\varphi_{G \circ f'_\diamond, f_\diamond}]{} & \underline{\mathbf{T}}(G \circ f'_\diamond) \circ \underline{\mathbf{T}}f_\diamond & \xrightarrow[\varphi_{G,f'_\diamond} * \underline{\mathbf{T}}f_\diamond]{} & \underline{\mathbf{T}}G \circ \underline{\mathbf{T}}f'_\diamond \circ \underline{\mathbf{T}}f_\diamond
\end{array}$$

   3.2 $\varphi_{f'_\diamond, f_\diamond} = \psi^{-1}_{f'_\diamond, f_\diamond}$, where $\psi_{f'_\diamond, f_\diamond} \colon \underline{\mathbf{T}}f'_\diamond \circ \underline{\mathbf{T}}f_\diamond \Rightarrow \underline{\mathbf{T}}(f'_\diamond \circ f_\diamond)$ is the lax associator for $\underline{\mathbf{T}}$.

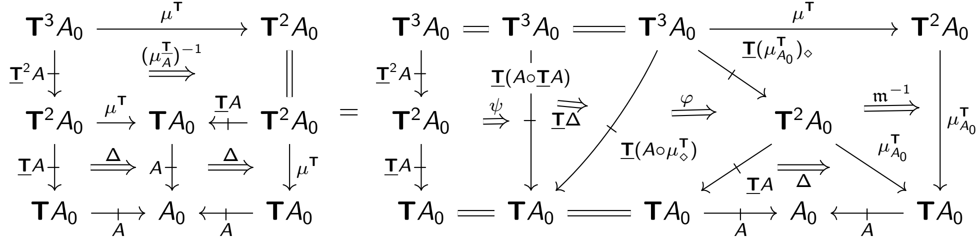
...then lax algebras are allowed

In essence, a left skew extension $\langle \underline{\mathbf{T}}, \eta^{\underline{\mathbf{T}}}, \mu^{\underline{\mathbf{T}}} \rangle$ allowing for lax algebras is *really* close to being a pseudomonad, enough so that we can define the 2-category $\mathbf{LaxAlg}_{r,co}(\underline{\mathbf{T}})$ of:

- lax $\underline{\mathbf{T}}$-algebras;



$\mathbf{T}A_0$
A_0
A
$\eta^{\mathbf{T}}$
$(\eta^{\mathbf{T}})^{-1}$
$\mathfrak{l}$
$\mathbf{T}^2A_0$
$\underline{\mathbf{T}}A$
Γ
$\mu^{\mathbf{T}}$
Δ
$=$
1_A
ψ
$\underline{\mathbf{T}}1_{A_0}$
$\underline{\mathbf{T}}\Gamma$
$\underline{\mathbf{T}}(A\circ(\eta^{\mathbf{T}})_\diamond)$
$\underline{\mathbf{T}}\eta^{\mathbf{T}}_\diamond$
φ
$\mathfrak{r}^{-1}$

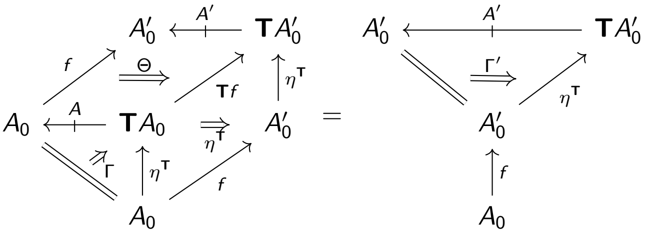


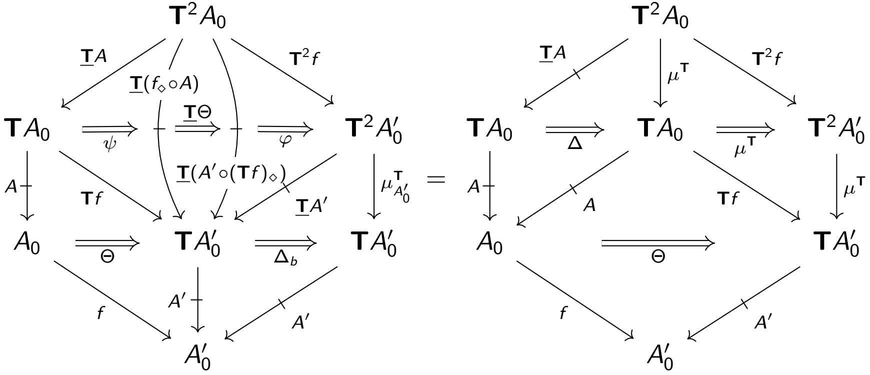
$\mathbf{T}^3A_0$
$\mathbf{T}^2A_0$
$\mathbf{T}A_0$
A_0
$\mu^{\mathbf{T}}$
$(\mu^{\mathbf{T}}_A)^{-1}$
$\underline{\mathbf{T}}^2A$
$\underline{\mathbf{T}}A$
A
Δ
$=$
$\underline{\mathbf{T}}(A\circ\underline{\mathbf{T}}A)$
ψ
$\underline{\mathbf{T}}\Delta$
$\underline{\mathbf{T}}(\mu^{\mathbf{T}}_{A_0})_\diamond$
$\underline{\mathbf{T}}(A\circ\mu^{\mathbf{T}}_\diamond)$
φ
$\mathfrak{m}^{-1}$
$\mu^{\mathbf{T}}_{A_0}$

...then lax algebras are allowed

In essence, a left skew extension $\langle \underline{\mathbf{T}}, \eta^{\underline{\mathbf{T}}}, \mu^{\underline{\mathbf{T}}} \rangle$ allowing for lax algebras is *really* close to being a pseudomonad, enough so that we can define the 2-category $\mathbf{LaxAlg}_{r,co}(\underline{\mathbf{T}})$ of:

- lax $\underline{\mathbf{T}}$-algebras;
- representable colax morphisms;

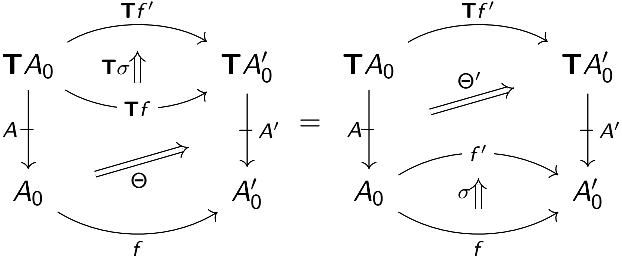

A'_0
A'
TA'_0
f
Θ
Tf
η^T
A_0
A
TA_0
A'_0
=
Γ'
Γ
f
A_0


T^2A_0
TA
T(f_◇∘A)
T^2f
TA_0
ψ
TΘ
φ
T^2A'_0
T(A'∘(Tf)_◇)
A
Tf
TA'
μ^T_{A'_0}
=
Δ
μ^T
A_0
Θ
TA'_0
Δ_b
f
A'
A'_0

...then lax algebras are allowed

In essence, a left skew extension $\langle \underline{\mathbf{T}}, \eta^{\underline{\mathbf{T}}}, \mu^{\underline{\mathbf{T}}} \rangle$ allowing for lax algebras is *really* close to being a pseudomonad, enough so that we can define the 2-category $\mathbf{LaxAlg}_{r,co}(\underline{\mathbf{T}})$ of:

- lax $\underline{\mathbf{T}}$-algebras;
- representable colax morphisms;
- algebra 2-cells.



$\mathbf{T}f'$
$\mathbf{T}A_0$
$\mathbf{T}\sigma\Uparrow$
$\mathbf{T}A_0'$
$\mathbf{T}f$
A
A'
Θ
A_0
A_0'
f
$=$
$\mathbf{T}f'$
$\mathbf{T}A_0$
$\mathbf{T}A_0'$
Θ'
A
A'
f'
A_0
$\sigma\Uparrow$
A_0'
f

...then lax algebras are allowed

In essence, a left skew extension $\langle \underline{\mathbf{T}}, \eta^{\underline{\mathbf{T}}}, \mu^{\underline{\mathbf{T}}} \rangle$ allowing for lax algebras is *really* close to being a pseudomonad, enough so that we can define the 2-category $\mathbf{LaxAlg}_{r,co}(\underline{\mathbf{T}})$ of:

- lax $\underline{\mathbf{T}}$-algebras;
- representable colax morphisms;
- algebra 2-cells.

...then lax algebras are allowed

In essence, a left skew extension $\langle \underline{\mathbf{T}}, \eta^{\underline{\mathbf{T}}}, \mu^{\underline{\mathbf{T}}} \rangle$ allowing for lax algebras is *really* close to being a pseudomonad, enough so that we can define the 2-category $\mathbf{LaxAlg}_{r,co}(\underline{\mathbf{T}})$ of:

- lax $\underline{\mathbf{T}}$-algebras;
- representable colax morphisms;
- algebra 2-cells.

Proposition

The ultracompletion pseudomonad $\mathbb{B}$ allows for lax algebras, and there is an isomorphism of categories between:

1. *ultraconvergence spaces and continuous maps;*
2. *discrete lax $\underline{\mathbb{B}}$-algebras and representable colax morphisms.*

...then lax algebras are allowed

In essence, a left skew extension $\langle \underline{\mathbf{T}}, \eta^{\underline{\mathbf{T}}}, \mu^{\underline{\mathbf{T}}} \rangle$ allowing for lax algebras is *really* close to being a pseudomonad, enough so that we can define the 2-category $\mathbf{LaxAlg}_{r,co}(\underline{\mathbf{T}})$ of:

- lax $\underline{\mathbf{T}}$-algebras;
- representable colax morphisms;
- algebra 2-cells.

Proposition

The ultracompletion pseudomonad $\mathbb{B}$ allows for lax algebras, and there is an isomorphism of categories between:

1. *ultraconvergence spaces and continuous maps;*
2. *discrete lax $\underline{\mathbb{B}}$-algebras and representable colax morphisms.*

- $\mathbf{LaxAlg}^{disc}_{r,co}(\underline{\mathbb{B}})$ is locally discrete!

Categories of points

To address this, we draw inspiration from the theory of generalized multicategories.

Definition

For a lax $\underline{\beta}$-algebra $A\colon \beta A_0 \nrightarrow A_0$, we define its category of points $(\mathrm{pt}\, A)_0$ as the category having:

- as objects, the objects of A_0;
- as morphisms $x \to y$, the elements of $A(x, (1, y, *))$.

Categories of points

To address this, we draw inspiration from the theory of generalized multicategories.

Definition

For a lax $\underline{\beta}$-algebra $A\colon \beta A_0 \nrightarrow A_0$, we define its category of points $(\mathrm{pt}\, A)_0$ as the category having:

- as objects, the objects of A_0;
- as morphisms $x \to y$, the elements of $A(x, (1, y, *))$.

A extends to a profunctor $\mathrm{pt}\, A\colon \beta(\mathrm{pt}\, A)_0 \nrightarrow (\mathrm{pt}\, A)_0$, also carrying the structure of a lax $\underline{\beta}$-algebra. The operation $A \mapsto \mathrm{pt}\, A$ is a closure operator on $\mathbf{LaxAlg}_{r,co}(\underline{\beta})$:

- $\mathrm{pt}(\mathrm{pt}\, A) \cong \mathrm{pt}\, A$;
- there is an identity-on-objects functor $\gamma_A\colon A_0 \to (\mathrm{pt}\, A)_0$ carrying the structure of a colax morphism $A \to \mathrm{pt}\, A$;

Normalization

Definition

A lax $\underline{\mathbb{\beta}}$-algebra $A\colon \mathbb{\beta} A_0 \to\!\!\!\!\!\!+\, A_0$ is normalized if γ_A is an isomorphism $A \cong \operatorname{pt} A$.

Normalization

Definition

A lax $\underline{\beta}$-algebra $A \colon \beta A_0 \twoheadrightarrow A_0$ is normalized if γ_A is an isomorphism $A \cong \operatorname{pt} A$.

At the level of objects and 1-cells, normalization is equivalent to discreteness.

Proposition

There are isomorphisms of categories between:

1. *ultraconvergence spaces and continuous maps;*
2. *discrete lax $\underline{\beta}$-algebras and representable colax morphisms;*
3. *normalized lax $\underline{\beta}$-algebras and representable colax morphisms.*

Normalization

Definition

A lax $\underline{\beta}$-algebra $A\colon \beta A_0 \to\!\!\!\!\!\!+\, A_0$ is normalized if γ_A is an isomorphism $A \cong \mathrm{pt}\, A$.

At the level of objects and 1-cells, normalization is equivalent to discreteness.

Proposition

There are isomorphisms of categories between:

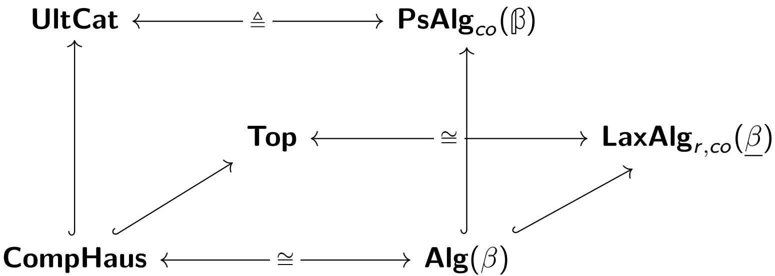
1. *ultraconvergence spaces and continuous maps;*
2. *discrete lax $\underline{\beta}$-algebras and representable colax morphisms;*
3. *normalized lax $\underline{\beta}$-algebras and representable colax morphisms.*

For 2-cells, instead, it makes all the difference: in the normalized description, algebra 2-cells recover the transformations between continuous maps.

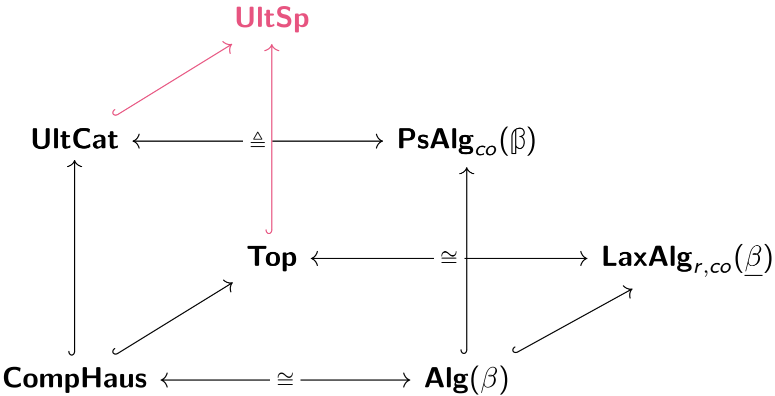
Theorem

There is an isomorphism of 2-categories $\mathbf{UltSp} \cong \mathbf{LaxAlg}^{norm}_{r,co}(\underline{\beta})$.

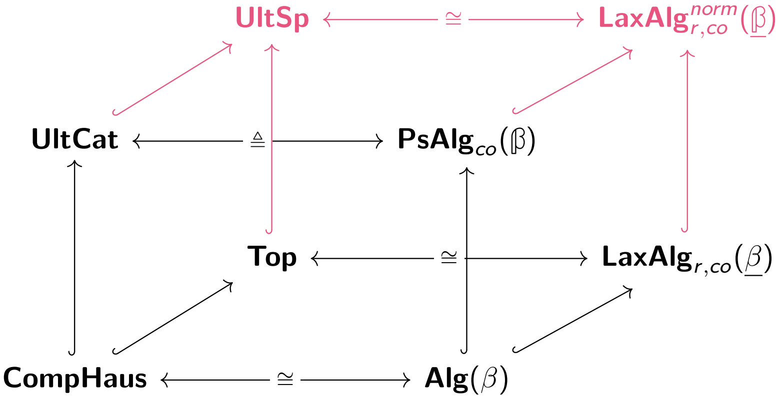
The big picture



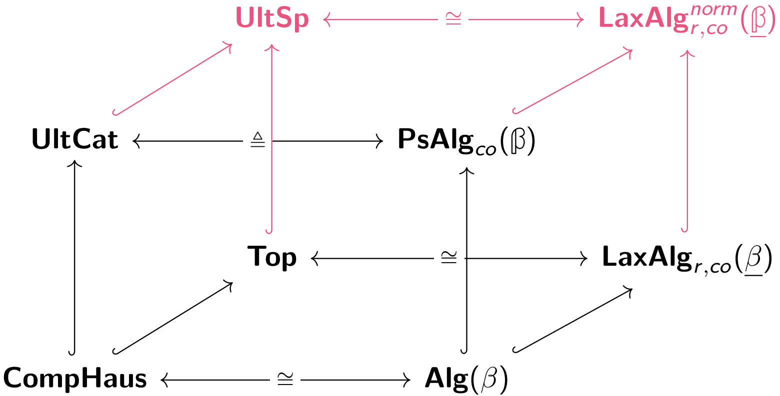
The big picture



The big picture



The big picture



Thank you!

E. G. Manes (1969). "A triple theoretic construction of compact algebras". In: *Seminar on Triples and Categorical Homology Theory*. Ed. by B. Eckmann. Vol. 80. Lecture notes in Mathematics. Berlin, Heidelberg: Springer, pp. 91–118.

M. Barr (1970). "Relational algebras". In: *Reports of the Midwest Category Seminar IV*. Ed. by S. MacLane et al. Berlin, Heidelberg: Springer, pp. 39–55.

M. Makkai (1987). "Stone duality for first order logic". In: *Advances in Mathematics* 65.2, pp. 97–170.

J. Lurie (2018). *Ultracategories*. Available online.

R. Garner (2020). "Ultrafilters, finite coproducts and locally connected classifying toposes". In: *Annals of Pure and Applied Logic* 171.10, p. 102831.

G. Rosolini (2024). *Ultracompletions*. Talk at CT2024.

G. Saadia (2025). *Extending conceptual completeness via virtual ultracategories*. arXiv: 2506.23935.

A. Hamad (2025). *Generalised ultracategories and conceptual completeness of geometric logic*. arXiv: 2507.07922.

Q. Aristote and U. Tarantino (2026). *Profunctorial algebras*. arXiv: 2601.22721.

S. van Gool, J. Marquès, and U. Tarantino (2026). "Toposes with enough points as categories of étale spaces". In: *Journal of Pure and Applied Algebra* 230.6, p. 108269.