

Toposes with enough points as categories of étale spaces

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Gothenburg Logic Seminar

26th May 2026

Introduction and motivation

Barr's *Relational algebras*

In 1970, Barr discusses how to extend a functor $T: \mathbf{Set} \rightarrow \mathbf{Set}$ to relations by seeing a relation $R \subseteq A \times B$ as the jointly-monic span of its projections $\langle q, p \rangle: R \rightrightarrows A \times B$,

$$\underline{T} \left(A \xleftarrow{q} R \xrightarrow{p} B \right) \triangleq TA \xrightarrow{+} TR \xrightarrow{+} TB$$

where, for a function $f: X \rightarrow Y$:

- ▶ $f_{\diamond} \subseteq X \times Y$ is the *graph* relation, $\{ (x, y) \mid f(x) = y \}$;
- ▶ $f^{\diamond} \subseteq Y \times X$ is the *cograph* relation, $\{ (y, x) \mid y = f(x) \}$,

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Proposition (if you believe in the Axiom of Choice)

The extension $\underline{T}: \mathbf{Rel} \rightarrow \mathbf{Rel}$ is an oplax functor:

$$\begin{array}{ccc}
 TA & \xrightarrow{T(1_A)} & TA \\
 \text{in} & & \\
 \text{---} & & \text{---} \\
 TA & & TA
 \end{array}
 \qquad
 \begin{array}{ccc}
 TA & \xrightarrow{T(S \circ R)} & TC \\
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Proposition (if you don't believe in the Axiom of Choice)

If T preserves surjections, the extension $\underline{T}: \mathbf{Rel} \rightarrow \mathbf{Rel}$ is an oplax functor:

$$\begin{array}{ccc}
 & \xrightarrow{\underline{T}(1_A)} & \\
 TA & \xrightarrow{\quad | \quad} & TC \\
 & \text{In} & \\
 & \xrightarrow{\quad | \quad} & \\
 & \xrightarrow{\underline{T}R} & \xrightarrow{\underline{T}S} \\
 & TB &
 \end{array}$$

Barr's Relational algebras

Extensions of natural transformations $\sigma: T \Rightarrow T'$ can be treated similarly.

Proposition

The extension $\underline{\sigma}: \underline{T} \Rightarrow \underline{T}'$ defined by $\underline{\sigma}_A = (\sigma_A)_\diamond$ is an oplax natural transformation.

$$\begin{array}{ccc} TA & \xrightarrow{\sigma_A} & T'A \\ \underline{T}R \downarrow & \subsetneq & \downarrow \underline{T}'R \\ TB & \xrightarrow{\sigma_B} & T'B \end{array}$$

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Combining these results, we obtain an extension result for **Set**-monads.

Theorem

Every **Set**-monad $\langle T, \eta^T, \mu^T \rangle$ extends to a *right skew monad* $\langle \underline{T}, \eta^{\underline{T}}, \mu^{\underline{T}} \rangle$ on **Rel**, a monad-like structure defined by oplax functors and oplax natural transformations.

Barr's *Relational algebras*

Barr's main motivation to study these extensions was to recover topological spaces as *relational algebras* for the *ultrafilter monad* $\beta: \mathbf{Set} \rightarrow \mathbf{Set}$.

Theorem

The category **Top** is isomorphic to the category $\mathbf{LaxAlg}_{r,co}(\underline{\beta})$ of *lax algebras* and *representable colax morphisms* for the skew monad $\underline{\beta}: \mathbf{Rel} \rightarrow \mathbf{Rel}$.

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Concretely, this means that:

- ▶ a topology on a set X can be equivalently described by a relation $\xi \subseteq \beta X \times X$ such that

$$\begin{array}{ccc}
 X & \xrightarrow{\quad \quad} & X \\
 \eta_X \searrow & \sqcap & \nearrow \xi \\
 & \beta X &
 \end{array}$$

$$\begin{array}{ccc}
 \beta^2 X & \xrightarrow{\beta\xi} & \beta X \\
 \mu_X \downarrow & \cong & \downarrow \xi \\
 \beta X & \xrightarrow{\xi} & X
 \end{array}$$

- ▶ a map $f: X \rightarrow X'$ is continuous if and only if

$$\begin{array}{ccc}
 \beta X & \xrightarrow{\beta f} & \beta X' \\
 \xi \downarrow & \subseteq & \downarrow \xi' \\
 X & \xrightarrow{f} & X
 \end{array}$$

Barr's *Relational algebras*

Convergence of ultrafilters

We can think of $\xi \subseteq \beta X \times X$ as specifying a notion of **convergence**:

- ▶ $\nu \xi x \iff \forall U \subseteq X \text{ open, if } x \in U \text{ then } U \in \nu$;
- ▶ $U \subseteq X \text{ open} \iff \forall \nu \in \beta X, x \in U, \text{ if } \nu \xi x \text{ then } U \in \nu$.

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For a function $f: I \rightarrow X$ and $\nu \in \beta I$, we write $x \rightsquigarrow_{i:\nu} f_i$ in case $\beta f(\nu) \xi x$ holds:

$$x \rightsquigarrow_{i:\nu} f_i \equiv \text{“}x \text{ is a limit of } f \text{ along } \nu\text{”}$$

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For a function $f: I \rightarrow X$ and $\nu \in \beta I$, we write $x \prec_{i:\nu} f_i$ in case $\beta f(\nu) \xi x$ holds:

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Restricting to *functional* convergence relations, we recover:

- ▶ topologically, compact Hausdorff spaces;
- ▶ algebraically, (ordinary) β -algebras,

i.e. Manes' theorem $\mathbf{CompHaus} \cong \mathbf{Alg}(\beta)$.

$$\begin{array}{ccc} \mathbf{Top} & \leftarrow \cong \rightarrow & \mathbf{LaxAlg}_{r,co}(\underline{\beta}) \\ \uparrow & & \uparrow \\ \mathbf{CompHaus} & \leftarrow \cong \rightarrow & \mathbf{Alg}(\beta) \end{array}$$

Now: one dimension higher!

Ultracategories

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For first-order logic, the role of Lindenbaum-Tarski algebras is played by **classifying toposes**: theories with equivalent classifying toposes have essentially the same models.

Theorem (Makkai; Lurie)

For a coherent theory \mathbb{T} , the category $\mathbf{Mod}(\mathbb{T})$ is an ultracategory, and $\mathbf{UltCat}(\mathbf{Mod}(\mathbb{T}), \mathbf{Set})$ is the classifying topos of \mathbb{T} .

Ultracategories

At its core, an ultracategory is a category C equipped with abstract **ultraproducts**, i.e. a functorial choice of a C -object $\prod_{i \in I} c_i$ for each **ultrafamily** in C , the datum of

- ▶ a set I ,
- ▶ an I -indexed family of C -objects $(c_i)_{i \in I}$, and
- ▶ an ultrafilter $\nu \in \beta I$,

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Ultracategories categorify compact Hausdorff spaces

CompHaus \hookrightarrow **UltCat** as the small and discrete ultracategories.

Ultracategories and coherent toposes

Identifying coherent theories with coherent toposes, and restricting to the subcategory \mathbf{UltCat}_* of ultracategories C such that $\mathbf{UltCat}(C, \mathbf{Set})$ is a topos, we have:

$$\begin{array}{ccc} & \mathbf{UltCat}(-, \mathbf{Set}) & \\ \leftarrow & \text{---} & \rightarrow \\ \mathbf{Topos}_{coh} & \perp & \mathbf{UltCat}_* \\ \leftarrow & \text{---} & \rightarrow \\ & \mathbf{pt}(-) & \end{array}$$

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$$\mathbf{Set}' \xrightarrow{\prod_{i:\nu} (-)} \mathbf{Set}$$

are **coherent**, i.e. they preserve finite limits, regular epis, and finite unions of subobjects.

What about geometric logic?

We will now consider **geometric logic**: a theory is geometric if its axioms are of the form $\forall \vec{x}(\varphi(\vec{x}) \rightarrow \psi(\vec{x}))$ where φ, ψ are built only using finitary \wedge , **infinitary** \vee , and \exists . For a geometric theory, Łoś's theorem fails: the ultraproduct functors are not **geometric**, as they don't preserve arbitrary unions of subobjects.

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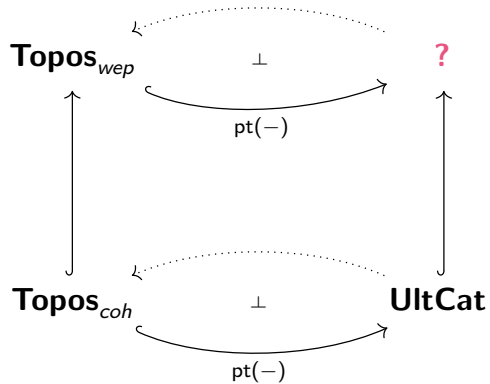
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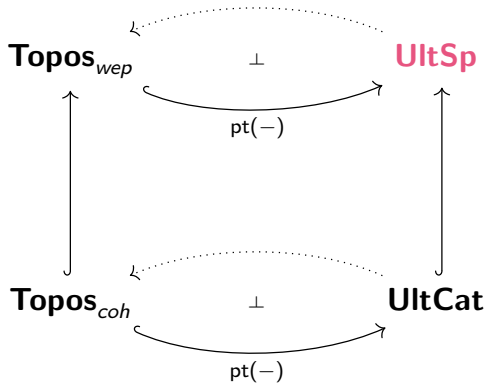
A necessary restriction

Having such a result, for a geometric theory \mathbb{T} , entails its completeness with respect to its **Set**-models. Categorically, this corresponds to restricting to toposes **with enough points**, a condition analogous to *spatiality* for locales.

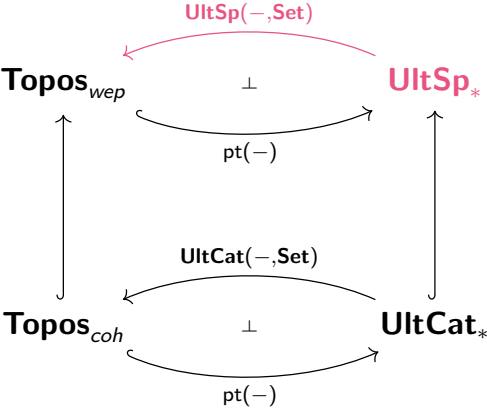
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Reconstruction via ultraconvergence spaces

Ultraconvergence spaces

The key intuition to address this question comes from Barr's theorem: we can generalize ultracategories by replacing a β -algebra functor with a **profunctor**, analogue of a relation between categories, and categorifying the notion of convergence.

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Definition

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- ▶ for every $x \in X$, an *identity* ultra-arrow $\text{id}_x: x \multimap_{*:1} x$;
- ▶ for every ultra-arrow $r: x \multimap_{i:\nu} y_i$ and every ultrafamily of ultra-arrows $(s_i: y_i \multimap_{j:\zeta_i} z_{i,j})_{i:\nu}$, a *composite* ultra-arrow $(s_i)_{i:\nu} \cdot r: x \multimap_{(i,j):\sum_{i:\nu} \zeta_i} z_{i,j}$,

satisfying equational axioms which express unitality and associativity of composition:

1. $(r)_{*:1} \cdot \text{id}_x = r$,
2. ...

Continuous maps

Similarly, we can extend the notion of continuity to this **Set**-valued convergence relation, which now becomes *structure* rather than *property*.

A **continuous map** of ultraconvergence spaces is a functor $f: X \rightarrow X'$ together with a mapping $(r: x \multimap_{i;\nu} y_i) \mapsto (f(r): f(x) \multimap_{i;\nu} f(y_i))$ on ultra-arrows, also satisfying equational axioms:

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A **transformation** of continuous maps $\alpha: f \Rightarrow f'$ is a family of ultra-arrows $\alpha_x: f(x) \multimap_{*:1} f'(x)$ in X' for each $x \in X$, such that for each ultra-arrow $r: x \multimap_{i:\nu} x_i$:

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UltSp is the 2-category of ultraconvergence spaces, continuous maps, and transformations.

Examples

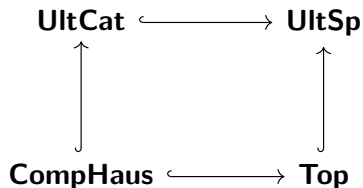
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- ▶ Every ultracategory C , defined by a functor $\Phi: \beta C \rightarrow C$, is an ultraconvergence space by setting ultra-arrows $c \dashrightarrow_{i;\nu} d_i$ to be C -arrows $c \rightarrow \Phi(I, d, \nu)$.

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The main theorem

As promised, the notion of ultraconvergence space allows us to obtain a reconstruction theorem for geometric logic: in topos-theoretical terms, it reads as follows.

Theorem (Saadia; Hamad; van Gool, Marquès, T.)

If \mathcal{E} is a topos with enough points, then $\mathcal{E} \simeq \mathbf{UltSp}(\text{pt}(\mathcal{E}), \mathbf{Set})$.

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In other words, restricting to the subcategory \mathbf{UltSp}_* of ultraconvergence spaces X such that $\mathbf{UltSp}(X, \mathbf{Set})$ is a topos, we have what we wanted:

$$\begin{array}{ccc} \mathbf{Topos}_{wep} & \begin{array}{c} \xleftarrow{\mathbf{UltSp}(-, \mathbf{Set})} \\ \perp \\ \xrightarrow{\text{pt}(-)} \end{array} & \mathbf{UltSp}_* \\ \updownarrow & & \updownarrow \\ \mathbf{Topos}_{coh} & \begin{array}{c} \xleftarrow{\mathbf{UltCat}(-, \mathbf{Set})} \\ \perp \\ \xrightarrow{\text{pt}(-)} \end{array} & \mathbf{UltCat}_* \end{array}$$

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Differently than Saadia's and Hamad's approaches, our proof crucially relies on a Grothendieck correspondence for continuous maps towards **Set**, which yields an extension of **local homeomorphisms** from topological to ultraconvergence spaces.

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2. for each $e \in E$ and each ultra-arrow $r: \pi(e) \multimap_{i;\nu} b_i$ in B , there is a unique *lift* $\bar{r}: e \multimap_{i;\nu} e_i$ in E such that $\pi(\bar{r}) = r$.

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$$\begin{array}{ccc} e & & \\ \pi \downarrow & & \\ \pi(e) & \xrightarrow{r} & (b_i)_{i:\nu} \end{array}$$

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Étale maps over B form a category **Et**(B), equivalent to **UltSp**(B , **Set**).

The ultraconvergence space of points of a topos

Let \mathcal{E} be a topos with a fixed class of points $X \subseteq \text{pt}(\mathcal{E})$.

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- ▶ For every object $\varphi \in \mathcal{E}$, we can define an étale space $\pi_\varphi: \llbracket \varphi \rrbracket \longrightarrow X$ where:
 - ▶ the fiber of π_φ at $x \in X$ is given by $x(\varphi)$;
 - ▶ an ultra-arrow $(x, \nu) \multimap_{i:\nu} (y_i, w_i)$ in $\llbracket \varphi \rrbracket$ is given by an ultra-arrow $r: x \multimap_{i:\nu} y_i$ in X such that $r_\varphi(\nu) = (w_i)_{i:\nu}$.

This assignment defines the *evaluation functor* $\llbracket - \rrbracket: \mathcal{E} \longrightarrow \text{Et}(X)$.

Reconstruction for geometric logic

Theorem

If X is a separating set of points of \mathcal{E} , then $\llbracket - \rrbracket: \mathcal{E} \longrightarrow \text{Et}(X)$ is an equivalence.

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Theorem

*Let \mathbb{T} be a geometric theory which is complete with respect to its **Set**-models. Then, $\mathbf{Mod}(\mathbb{T})$ is an ultraconvergence space by setting ultra-arrows $M \dashv_{i:\nu} N_i$ to be structure morphisms $M \rightarrow \prod_{i:\nu} N_i$, and $\mathbf{Et}(\mathbf{Mod}(\mathbb{T}))$ is the classifying topos of \mathbb{T} .*

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The localic/propositional case

In particular, if a localic topos \mathcal{E} has enough points, i.e. $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{O}(X))$ for some topological space X , then $\mathcal{E} \simeq \mathbf{Et}(X)$.

Proof sketch

Our proof proceeds similarly to Makkai's original work, in two main steps.

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1. $\llbracket - \rrbracket : \mathcal{E} \rightarrow \text{Et}(X)$ is **full on subobjects**: every subobject of $\pi_\varphi : \llbracket \varphi \rrbracket \rightarrow X$ in $\text{Et}(X)$ is the restriction of π_φ to $\llbracket \psi \rrbracket \subseteq \llbracket \varphi \rrbracket$ for some subobject $\psi \rightarrow \varphi$ in \mathcal{E} .

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2. $\llbracket - \rrbracket: \mathcal{E} \rightarrow \text{Et}(X)$ is **covering**: every étale space $p: Y \rightarrow X$ is covered by an epimorphism $\alpha: \pi_\varphi \twoheadrightarrow p$ in $\text{Et}(X)$ for some object $\varphi \in \mathcal{E}$.

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Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $\llbracket - \rrbracket$.

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Two points of view

Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $\llbracket - \rrbracket$. However, we can also interpret (1) as stating that $\llbracket - \rrbracket$ defines a hyperconnected geometric morphism, and (2) as stating that it defines a localic geometric morphism.

Ultraconvergence spaces as profunctorial algebras

Extensions *à la* Barr

As it turns out, the inspiration from Barr's theorem can be pushed even further, to a 2-dimensional analogous extension theorem.

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For our aims, it's simpler to represent profunctors via *cospans* rather than spans.

Fact

Profunctors $C \rightrightarrows D$ can be identified with certain cospans $C \rightarrow V \leftarrow D$ called *two-sided codiscrete cofibrations*. Concretely:

- ▶ for $F: C \rightrightarrows D$, its *collage* is the cospan $C \xrightarrow{\iota_C} V \xleftarrow{\iota_D} D$ where V is the category having
 - as objects, the disjoint union of the objects of C and D ;
 - as morphisms, the morphisms of C and D together with a morphism $x: d \rightarrow c$ for each $d \in D$, $c \in C$, and $x \in F(d, c)$,

with the evident inclusions ι_C and ι_D ;

- ▶ for $C \xrightarrow{q} V \xleftarrow{p} D$, the profunctor $F: C \rightrightarrows D$ is defined by $F(d, c) = V(p(d), q(c))$.

Extensions à la Barr

The idea is then to study how to extend a pseudofunctor $\mathbf{T}: \mathbf{CAT} \rightarrow \mathbf{CAT}$ to profunctors by identifying them with their collages,

$$\mathbf{T} \left(C \xrightarrow{\iota_C} E \xleftarrow{\iota_D} D \right) \triangleq \mathbf{T}C \xrightarrow{(\mathbf{T}\iota_C)_\diamond} \mathbf{T}E \xrightarrow{(\mathbf{T}\iota_D)^\diamond} \mathbf{T}D$$

where, for a functor $f: X \rightarrow Y$:

- ▶ $f_\diamond: X \rightrightarrows Y$ is the *graph* profunctor, $f_\diamond(y, x) = Y(y, f(x))$;
- ▶ $f^\diamond: Y \rightrightarrows X$ is the *cograph* profunctor, $f^\diamond(x, y) = Y(f(x), y)$,

giving rise to locally fully-faithful pseudofunctors:

$$(-)_\diamond: \mathbf{CAT} \rightarrow \mathbf{PROF} \quad (-)^\diamond: \mathbf{CAT}^{\text{op}} \rightarrow \mathbf{PROF}^{\text{co}}$$

Extending pseudofunctors

Barr's preservation of surjections here becomes preservation of fully-faithful functors.

Proposition

If \mathbf{T} preserves fully-faithfulness, the extension $\underline{\mathbf{T}}: \mathbf{PROF} \rightarrow \mathbf{PROF}$ is a lax functor:

$$\begin{array}{ccc}
 \mathbf{T}C & \xrightarrow{\mathbf{T}1_C} & \mathbf{T}C \\
 \psi_C \uparrow & & \uparrow \\
 \mathbf{T}C & & \mathbf{T}C
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbf{T}C & \xrightarrow{\mathbf{T}(G \circ F)} & \mathbf{T}E \\
 \downarrow \mathbf{T}F & \psi_{G,F} \uparrow & \uparrow \mathbf{T}G \\
 & \mathbf{T}D &
 \end{array}$$

Moreover, $\underline{\mathbf{T}}$ is equipped with a lax natural transformation $\delta^{\mathbf{T}}: \underline{\mathbf{T}}(-)_{\diamond} \Rightarrow (\mathbf{T}-)_{\diamond}$:

$$\begin{array}{ccc}
 \mathbf{PROF} & \xrightarrow{\underline{\mathbf{T}}} & \mathbf{PROF} \\
 (-)_{\diamond} \uparrow & \delta^{\mathbf{T}} \swarrow & \uparrow (-)_{\diamond} \\
 \mathbf{CAT} & \xrightarrow{\mathbf{T}} & \mathbf{CAT}
 \end{array}$$

Extending transformations and modifications

Extensions of pseudonatural transformations $\sigma: \mathbf{T} \Rightarrow \mathbf{T}'$ and modifications $\mathfrak{m}: \sigma \Rrightarrow \sigma'$ can be treated similarly, simply by carrying them along $(-)_\diamond: \mathbf{CAT} \rightarrow \mathbf{PROF}$.

Proposition

The extension $\underline{\sigma}: \underline{\mathbf{T}} \Rightarrow \underline{\mathbf{T}'}$ defined by $\underline{\sigma}_C = (\sigma_C)_\diamond$ is a lax natural transformation.

$$\begin{array}{ccc} \mathbf{T}C & \xrightarrow{\sigma_C} & \mathbf{T}'C \\ \underline{\mathbf{T}}F \downarrow & \swarrow \sigma_F & \downarrow \underline{\mathbf{T}'}F \\ \mathbf{T}D & \xrightarrow{\sigma_D} & \mathbf{T}'D \end{array}$$

Proposition

The extension $\underline{\mathfrak{m}}: \underline{\sigma} \Rrightarrow \underline{\sigma}'$ defined by $\underline{\mathfrak{m}}_C = (\mathfrak{m}_C)_\diamond$ is a modification.

Extending pseudomonads

Combining these results, we obtain an extension result for pseudomonads on **CAT**.

Theorem (Aristote, T.)

Every pseudomonad $\langle \mathbf{T}, \eta^{\mathbf{T}}, \mu^{\mathbf{T}} \rangle$ on **CAT** preserving fully-faithfulness extends to a *left skew monad* $\langle \underline{\mathbf{T}}, \eta^{\underline{\mathbf{T}}}, \mu^{\underline{\mathbf{T}}} \rangle$ on **PROF**.

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In particular, mimicking Barr, we can apply it to β .

Corollary

The ultracompletion pseudomonad β extends to a left skew monad $\underline{\beta}$ on **PROF**.

Ultraconvergence spaces as discrete algebras

Fact

The structure of an ultraconvergence space on a discrete category X coincides with that of a *lax β -algebra* on X , i.e. a profunctor $\Xi: \beta X \rightarrow X$ together with two natural transformations:

$$\begin{array}{ccc}
 X & \xrightarrow{1_X} & X \\
 \eta_X \searrow & \Downarrow \Gamma & \nearrow \Xi \\
 & \beta X &
 \end{array}
 \qquad
 \begin{array}{ccc}
 \beta^2 X & \xrightarrow{\beta \Xi} & \beta X \\
 \mu_X \downarrow & \swarrow \Delta & \downarrow \Xi \\
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Fact

The structure of a continuous map on a functor $f: X \rightarrow X'$ coincides with that of a *colax morphism of lax β -algebras* on f , i.e. a natural transformation:

$$\begin{array}{ccc}
 \beta X & \xrightarrow{\beta f} & \beta X' \\
 \Xi \downarrow & \nearrow \Theta & \downarrow \Xi' \\
 X & \xrightarrow{f} & X'
 \end{array}$$

Now what?

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Second problem

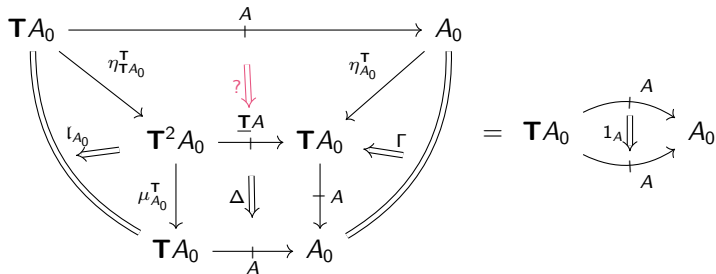
Algebra 2-cells, however defined, trivialize. How do transformations between continuous maps arise?

A coherence problem

Unlike the 1-dimensional case, here we have to consider appropriate coherence conditions which are not evident in this lax setting unless we assume more properties.

A concrete example

For a lax \mathbf{T} -algebra $A: \mathbf{T}A_0 \dashrightarrow A_0$, there should be a 'right unitality' axiom



but, being $\eta^{\mathbf{T}}$ lax natural, its structural 2-cell goes in the wrong way.

- For ultraconvergence spaces, this should read as $(r)_{*:1} \cdot \text{id}_x = r$ for all $r: x \dashrightarrow_{i:\nu} y_i$.

If we allow for lax algebras...

Definition

The skew extension $\langle \underline{\mathbf{T}}, \eta^{\mathbf{T}}, \mu^{\mathbf{T}} \rangle$ of a pseudomonad $\langle \mathbf{T}, \eta^{\mathbf{T}}, \mu^{\mathbf{T}} \rangle$ allows for lax algebras if:

1. the unit $\eta^{\mathbf{T}}$ and the multiplication $\mu^{\mathbf{T}}$ are pseudonatural;
2. the transformation $\delta^{\mathbf{T}}: \underline{\mathbf{T}}(-)_{\diamond} \Rightarrow (\mathbf{T}-)_{\diamond}$ is pseudonatural;
3. there exists a natural family of natural transformations $\varphi_{G, f_{\diamond}}: \underline{\mathbf{T}}(G \circ f_{\diamond}) \Rightarrow \underline{\mathbf{T}}G \circ \underline{\mathbf{T}}f_{\diamond}$, for any functor $f: A \rightarrow B$ and any profunctor $G: B \dashrightarrow C$, such that for each pair of functors $f: A \rightarrow B, f': B \rightarrow C$ and each profunctor $G: C \dashrightarrow D$:

- 3.1 the diagram below commutes;

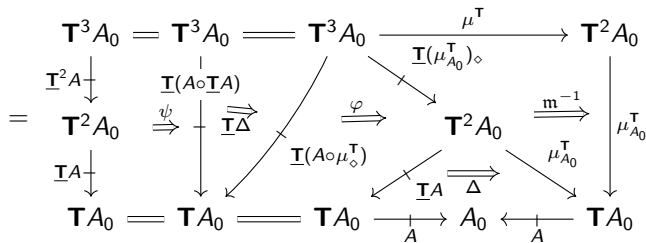
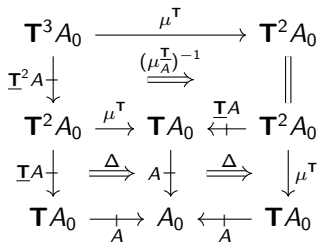
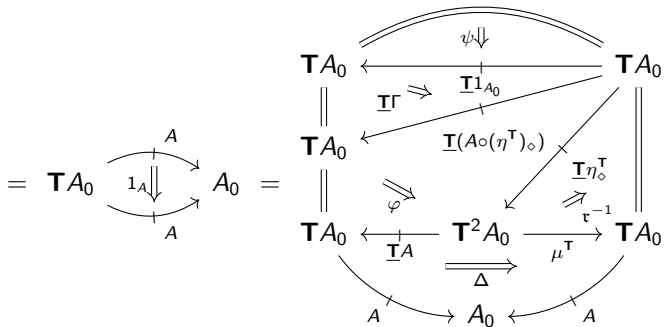
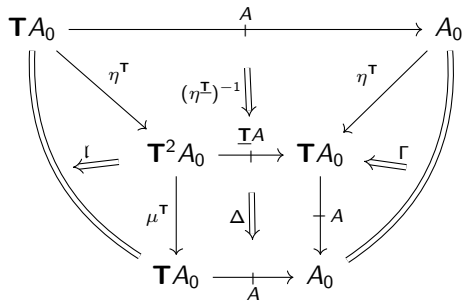
$$\begin{array}{ccccc}
 \underline{\mathbf{T}}(G \circ (f'f)_{\diamond}) & \xrightarrow{\varphi_{G, (f'f)_{\diamond}}} & \underline{\mathbf{T}}G \circ \underline{\mathbf{T}}(f'f)_{\diamond} & \xrightarrow{\cong} & \underline{\mathbf{T}}G \circ \underline{\mathbf{T}}(f'_{\diamond} \circ f_{\diamond}) \\
 \cong \downarrow & & & & \downarrow \underline{\mathbf{T}}G * \varphi_{f'_{\diamond}, f_{\diamond}} \\
 \underline{\mathbf{T}}(G \circ f'_{\diamond} \circ f_{\diamond}) & \xrightarrow{\varphi_{G \circ f'_{\diamond}, f_{\diamond}}} & \underline{\mathbf{T}}(G \circ f'_{\diamond}) \circ \underline{\mathbf{T}}f_{\diamond} & \xrightarrow{\varphi_{G, f'_{\diamond}} * \underline{\mathbf{T}}f_{\diamond}} & \underline{\mathbf{T}}G \circ \underline{\mathbf{T}}f'_{\diamond} \circ \underline{\mathbf{T}}f_{\diamond}
 \end{array}$$

- 3.2 $\varphi_{f'_{\diamond}, f_{\diamond}} = \psi_{f'_{\diamond}, f_{\diamond}}^{-1}$, where $\psi_{f'_{\diamond}, f_{\diamond}}: \underline{\mathbf{T}}f'_{\diamond} \circ \underline{\mathbf{T}}f_{\diamond} \Rightarrow \underline{\mathbf{T}}(f'_{\diamond} \circ f_{\diamond})$ is the lax associator for $\underline{\mathbf{T}}$.

...then lax algebras are allowed

In essence, a left skew extension $\langle \underline{\mathbf{T}}, \eta^{\underline{\mathbf{T}}}, \mu^{\underline{\mathbf{T}}} \rangle$ allowing for lax algebras is *really* close to being a pseudomonad, enough so that we can define the 2-category $\mathbf{LaxAlg}_{r,co}(\underline{\mathbf{T}})$ of:

- ▶ lax $\underline{\mathbf{T}}$ -algebras;



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- ▶ lax $\underline{\mathbf{T}}$ -algebras;
- ▶ representable colax morphisms;

$$\begin{array}{ccc}
 & A' & \\
 & \longleftarrow & \\
 & A'_0 & \longleftarrow \text{TA}'_0 \\
 \nearrow f & \xrightarrow{\Theta} & \nearrow \text{T}f \\
 A_0 & \longleftarrow \text{TA}_0 & \xrightarrow{\eta^\top} A'_0 \\
 \searrow \Gamma & \xrightarrow{\eta^\top} & \searrow f \\
 & A_0 &
 \end{array}
 =
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 & A' & \\
 & \longleftarrow & \\
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 \searrow \Gamma' & \xrightarrow{\eta^\top} & \searrow \eta^\top \\
 & A'_0 & \\
 \uparrow f & & \\
 & A_0 &
 \end{array}$$

$$\begin{array}{ccccc}
 & & \text{T}^2 A_0 & & \\
 & \swarrow \underline{\text{T}}A & & \searrow \text{T}^2 f & \\
 \text{TA}_0 & & & & \text{T}^2 A'_0 \\
 \xrightarrow{\psi} & \xrightarrow{\underline{\text{T}}\Theta} & & \xrightarrow{\varphi} & \\
 \downarrow A & \downarrow \text{T}f & & \downarrow \underline{\text{T}}A' & \downarrow \mu_{A'_0}^\top \\
 A_0 & \xrightarrow{\Theta} & \text{TA}'_0 & \xrightarrow{\Delta_b} & \text{TA}'_0 \\
 \searrow f & & \downarrow A' & \swarrow A' & \\
 & & A'_0 & &
 \end{array}
 =
 \begin{array}{ccccc}
 & & \text{T}^2 A_0 & & \\
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 \text{TA}_0 & & & & \text{T}^2 A'_0 \\
 \xrightarrow{\Delta} & \xrightarrow{\mu^\top} & & \xrightarrow{\mu^\top} & \\
 \downarrow A & \downarrow A & & \downarrow \text{T}f & \downarrow \mu^\top \\
 A_0 & \xrightarrow{\Theta} & \text{TA}'_0 & & \text{TA}'_0 \\
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- ▶ lax $\underline{\mathbf{T}}$ -algebras;
- ▶ representable colax morphisms;
- ▶ algebra 2-cells.

$$\begin{array}{ccc}
 \mathbf{T}A_0 & \xrightarrow{\mathbf{T}f'} & \mathbf{T}A'_0 \\
 \downarrow A & \begin{array}{c} \mathbf{T}\sigma \uparrow \\ \mathbf{T}f \end{array} & \downarrow A' \\
 A_0 & \xrightarrow{\Theta} & A'_0 \\
 \downarrow f & & \downarrow f
 \end{array}
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 \begin{array}{ccc}
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- ▶ representable colax morphisms;
- ▶ algebra 2-cells.

Proposition

The ultracompletion pseudomonad β allows for lax algebras, and there is an isomorphism of categories between:

- 1. ultraconvergence spaces and continuous maps;*
- 2. discrete lax β -algebras and representable colax morphisms.*

...then lax algebras are allowed

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Proposition

The ultracompletion pseudomonad $\underline{\beta}$ allows for lax algebras, and there is an isomorphism of categories between:

1. *ultraconvergence spaces and continuous maps;*
2. *discrete lax $\underline{\beta}$ -algebras and representable colax morphisms.*

- ▶ $\mathbf{LaxAlg}_{r,co}^{disc}(\underline{\beta})$ is locally discrete!

Categories of points

To address this, we draw inspiration from the theory of generalized multicategories.

Definition

For a lax $\underline{\beta}$ -algebra $A: \underline{\beta}A_0 \dashrightarrow A_0$, we define its **category of points** $(\text{pt } A)_0$ as the category having:

- ▶ as objects, the objects of A_0 ;
- ▶ as morphisms $x \rightarrow y$, the elements of $A(x, (1, y, *))$.

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A extends to a profunctor $\text{pt } A: \underline{\beta}(\text{pt } A)_0 \dashrightarrow (\text{pt } A)_0$, also carrying the structure of a lax $\underline{\beta}$ -algebra. The operation $A \mapsto \text{pt } A$ is a closure operator on $\mathbf{LaxAlg}_{r,co}(\underline{\beta})$:

- ▶ $\text{pt}(\text{pt } A) \cong \text{pt } A$;
- ▶ there is an identity-on-objects functor $\gamma_A: A_0 \rightarrow (\text{pt } A)_0$ carrying the structure of a colax morphism $A \rightarrow \text{pt } A$;

Normalization

Definition

A lax $\underline{\beta}$ -algebra $A: \underline{\beta}A_0 \rightarrow A_0$ is **normalized** if γ_A is an isomorphism $A \cong \text{pt } A$.

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At the level of objects and 1-cells, normalization is equivalent to discreteness.

Proposition

There are isomorphisms of categories between:

- 1. ultraconvergence spaces and continuous maps;*
- 2. discrete lax $\underline{\beta}$ -algebras and representable colax morphisms;*
- 3. normalized lax $\underline{\beta}$ -algebras and representable colax morphisms.*

Normalization

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Proposition

There are isomorphisms of categories between:

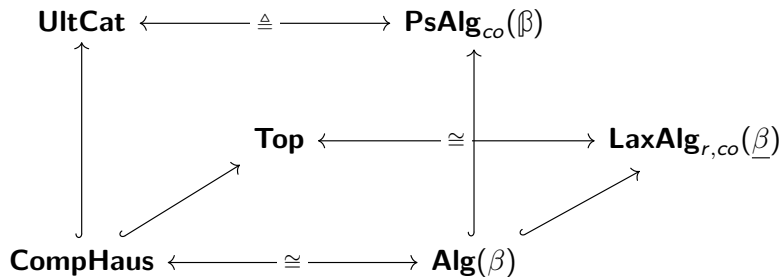
1. *ultraconvergence spaces and continuous maps;*
2. *discrete lax $\underline{\beta}$ -algebras and representable colax morphisms;*
3. *normalized lax $\underline{\beta}$ -algebras and representable colax morphisms.*

For 2-cells, instead, it makes all the difference: in the normalized description, algebra 2-cells recover the transformations between continuous maps.

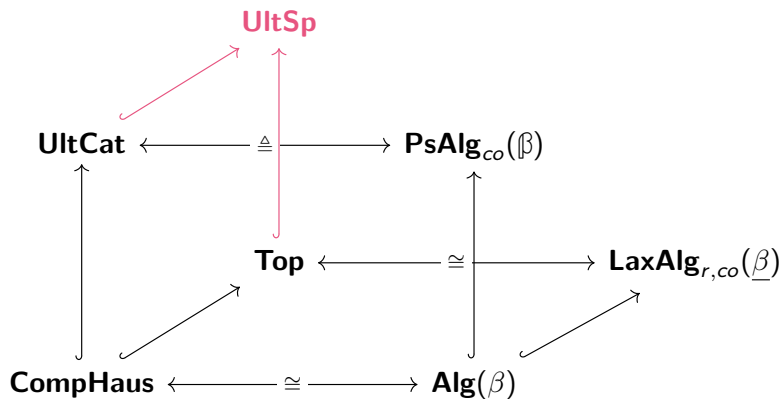
Theorem

There is an isomorphism of 2-categories $\mathbf{UltSp} \cong \mathbf{LaxAlg}_{r,co}^{norm}(\underline{\beta})$.

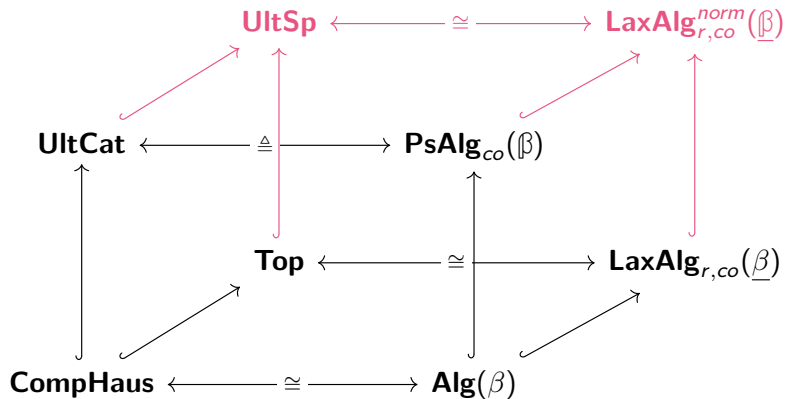
The big picture



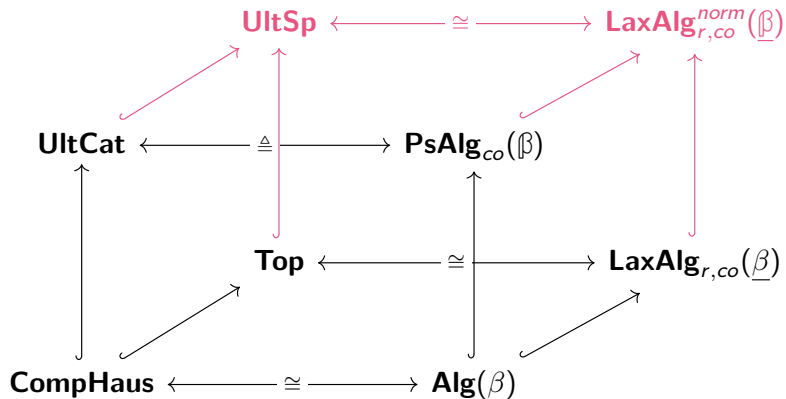
The big picture













The big picture



The big picture



Thank you!

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