2-dimensional topologies via profunctorial algebras

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Introduction and motivation

Barr's *Relational algebras*

In 1970, Barr discusses how to extend a functor $T\colon \mathbf{Set} \to \mathbf{Set}$ to relations by seeing a relation $R \subseteq A \times B$ as the jointly-monic span of its projections $\langle q, p \rangle : R \rightarrowtail A \times B$,

$$\underline{T}\left(A \xleftarrow{\;q\;} R \xrightarrow{\;p\;} B \right) \triangleq TA \xrightarrow{Tq^{\diamond}} TR \xrightarrow{Tp_{\diamond}} TB$$

where, for a function $f\colon X \to Y$:

- $f_{\diamond} \subseteq X \times Y$ is the *graph* relation, $\{\, (x, y) \mid f(x) = y \,\}$;
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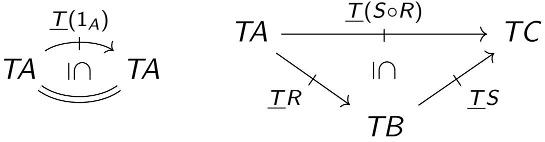
Proposition (if you believe in the Axiom of Choice)

The extension $\underline{T}\colon \mathbf{Rel} \to \mathbf{Rel}$ *is an oplax functor:*



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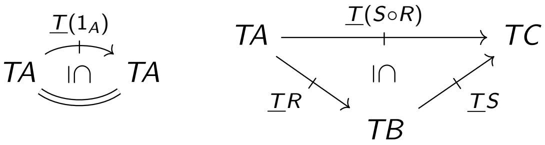
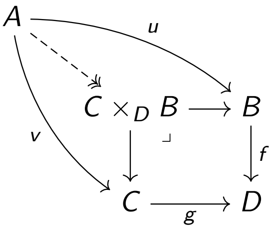
Proposition (if you don't believe in the Axiom of Choice)

*If T preserves surjections, the extension $\underline{T}\colon$ **Rel** $\to$ **Rel** is an oplax functor:*



Barr's *Relational algebras*

Moreover, he characterizes when this extension is a 2-functor in terms of *weak pullbacks*.



Definition

A commuting square $fu = gv$ in **Set** is a weak pullback if

$$\forall b \in B, c \in C \,.\, f(b) = g(c) \Rightarrow \exists a \in A : b = u(a) \wedge c = v(a)$$

i.e. if the universal function $A \to C \times_D B$ is surjective.

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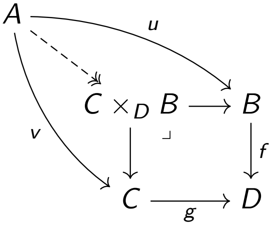

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Proposition

The extension $\underline{T}$: **Rel** $\to$ **Rel** *is a 2-functor if and only if* T *preserves weak pullbacks.*

Barr's *Relational algebras*

Extensions of natural transformations $\sigma\colon T \Rightarrow T'$ can be treated similarly.

Proposition

The extension $\underline{\sigma}\colon \underline{T} \Rightarrow \underline{T}'$ defined by $\underline{\sigma}_A = (\sigma_A)_\diamond$ is an oplax natural transformation, and it is strictly natural if and only if the naturality squares of σ are weak pullbacks.

$$\begin{array}{ccc} TA & \xrightarrow{\sigma_A} & T'A \\ {\scriptstyle \underline{T}R}\big\downarrow & \subseteq & \big\downarrow{\scriptstyle \underline{T}'R} \\ TB & \xrightarrow[\sigma_B]{} & T'B \end{array}$$

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Combining these results, we obtain an extension result for **Set**-monads.

Theorem

*Every **Set**-monad $\langle T, \eta^T, \mu^T \rangle$ preserving surjectivity extends to a right skew monad $\langle \underline{T}, \eta^{\underline{T}}, \mu^{\underline{T}} \rangle$ on **Rel**, which is a 2-monad if and only if:*

1. *T preserves weak pullbacks;*
2. *the naturality squares of η^T and μ^T are weak pullbacks.*

Barr's *Relational algebras*

Barr's main motivation to study these extensions was to recover topological spaces as relational algebras for the ultrafilter monad $\beta\colon \mathbf{Set} \to \mathbf{Set}$.

Theorem

The category **Top** *is isomorphic to the category* $\mathbf{LaxAlg}_{r,co}(\underline{\beta})$ *of lax algebras and representable colax morphisms for the skew monad* $\underline{\beta}\colon \mathbf{Rel} \to \mathbf{Rel}$*.*

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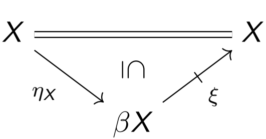
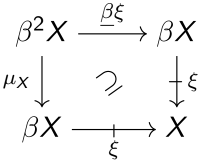
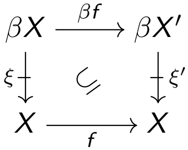
Concretely, this means that:

- a topology on a set X can be equivalently described by a relation $\xi \subseteq \beta X \times X$ such that





- a map $f\colon X \to X'$ is continuous if and only if



Barr's *Relational algebras*

Convergence of ultrafilters

We can think of $\xi \subseteq \beta X \times X$ as specifying a notion of convergence:

- $\nu \; \xi \; x \iff \forall U \subseteq X \text{ open, if } x \in U \text{ then } U \in \nu$;
- $U \subseteq X \text{ open} \iff \forall \nu \in \beta X, x \in U, \text{ if } \nu \; \xi \; x \text{ then } U \in \nu$.

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Restricting to functional convergence relations, we recover:

- topologically, compact Hausdorff spaces;
- algebraically, (ordinary) β-algebras,

i.e. Manes' theorem **CompHaus** $\cong$ **Alg**(β).

$$\begin{array}{ccc} \mathbf{Top} & \xleftarrow{\;\cong\;}\!\!\rightarrow & \mathbf{LaxAlg}_{r,co}(\underline{\beta}) \\ \uparrow & & \uparrow \\ \mathbf{CompHaus} & \xleftarrow{\;\cong\;}\!\!\rightarrow & \mathbf{Alg}(\beta) \end{array}$$

Two-dimensional topological spaces

In 1987, Makkai introduced ultracategories to prove a reconstruction theorem for first-order logic. At its core, an ultracategory is a category C endowed with abstract ultraproducts, i.e. a functorial choice of an object $\prod_{i:\nu} c_i$ in C for each set I, each I-indexed family of objects $(c_i)_{i \in I}$ in C, and each ultrafilter $\nu \in \beta I$, behaving as ultraproducts of sets.

Theorem (Makkai, Lurie)

For a coherent theory $\mathbb{T}$*, the category* $\mathbf{Mod}(\mathbb{T})$ *is an ultracategory, and* $\mathbf{UltCat}(\mathbf{Mod}(\mathbb{T}), \mathbf{Set})$ *is the classifying topos of* $\mathbb{T}$*.*

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Ultracategories categorify compact Hausdorff spaces

$\mathbf{CompHaus} \hookrightarrow \mathbf{UltCat}$ as the small and discrete ultracategories.

Two-dimensional topological spaces

In 2025, ultracategories were generalized to ultraconvergence spaces. At its core, an ultraconvergence space consists of a category C endowed with 'virtual arrows' $c \prec_{i:\nu} c_i$ into formal ultraproducts which might not exist in C. In particular, if C is an ultracategory, these are just actual arrows $c \to \prod_{i:\nu} c_i$.

Theorem (Saadia; Hamad; Van Gool, Marquès, T.)

For a geometric theory $\mathbb{T}$ which is complete with respect to its **Set***-models, the category* $\mathbf{Mod}(\mathbb{T})$ *is an ultraconvergence space, and* $\mathbf{UltSp}(\mathbf{Mod}(\mathbb{T}), \mathbf{Set})$ *is the classifying topos of* $\mathbb{T}$*.*

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Ultraconvergence spaces categorify topological spaces

$\mathbf{Top} \hookrightarrow \mathbf{UltSp}$ as (some) two-valued ultraconvergence spaces.

Two-dimensional topological spaces

To define ultracategories and ultraconvergence spaces formally, we need to introduce the ultracompletion pseudomonad $\mathbb{\beta}\colon \mathbf{CAT} \to \mathbf{CAT}$.

Definition (Garner, Rosolini)

For a category C, the category $\mathbb{\beta}C$ has:

- as objects, triples (I, y, ν) of a set I, a functor $y\colon I \to C$, and an ultrafilter $\nu \in \beta I$, which we refer to as ultrafamilies in C;
- as morphisms $(I, y, \nu) \to (I', y', \nu')$, pairs of a function $h\colon I' \to I$ such that $\beta h(\nu') = \nu$ and a family of arrows $(\alpha_i\colon y_{h(i)} \to y'_i)_{i \in I'}$ in C, both considered up to ν'-equivalence.

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Definition (Garner, Rosolini)

UltCat is the 2-category of pseudo-β-algebras, colax morphisms, and algebra 2-cells.

Two-dimensional topological spaces

Definition (Saadia; Hamad; Van Gool, Marquès, T.)

An ultraconvergence space consists of a discrete category X together with a profunctor $\Xi\colon X \times \beta X \longrightarrow \mathbf{Set}$, where elements of $\Xi(x, (I, y, \nu))$ are dubbed *ultra-arrows* and denoted by $r\colon x \prec_{i:\nu} y_i$. Moreover, X is equipped with:

- for every $x \in X$, an *identity* ultra-arrow $\mathrm{id}_x\colon x \prec_{*:1} x$;
- for every ultra-arrow $r\colon x \prec_{i:\mu} y_i$ and every ultrafamily of ultra-arrows $(s_i\colon y_i \prec_{j:\nu_i} z_{i,j})_{i:\mu}$, a *composite* ultra-arrow $(s_i)_{i:\mu} \cdot r\colon x \prec_{(i,j):\sum_{i:\mu}\nu_i} z_{i,j}$,

satisfying equational axioms which express unitality and associativity of composition:

1. $(r)_{*:1} \cdot \mathrm{id}_x = r$,
2. ...

Two-dimensional topological spaces

A continuous map of ultraconvergence spaces is a functor $f\colon X \longrightarrow X'$ together with a mapping $(r\colon x \prec_{i:\nu} y_i) \mapsto (f(r)\colon f(x) \prec_{i:\nu} f(y_i))$ on ultra-arrows, also satisfying equational axioms:

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A transformation of continuous maps $\alpha\colon f \Rightarrow f'$ is a family of ultra-arrows $\alpha_x\colon f(x) \prec_{*:1} f'(x)$ in X' for each $x \in X$, such that

$$f'(r) \cdot \alpha_x = (\alpha_{x_i})_{i:\mu} \cdot f(r)$$

for each ultra-arrow $r\colon x \prec_{i:\mu} x_i$ in X.

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UltSp is the 2-category of ultraconvergence spaces, continuous maps, and transformations.

The big picture

The aim of this talk is to present a 2-dimensional version of Barr's extension results, and to show how they can be used to recover ultraconvergence spaces algebraically.

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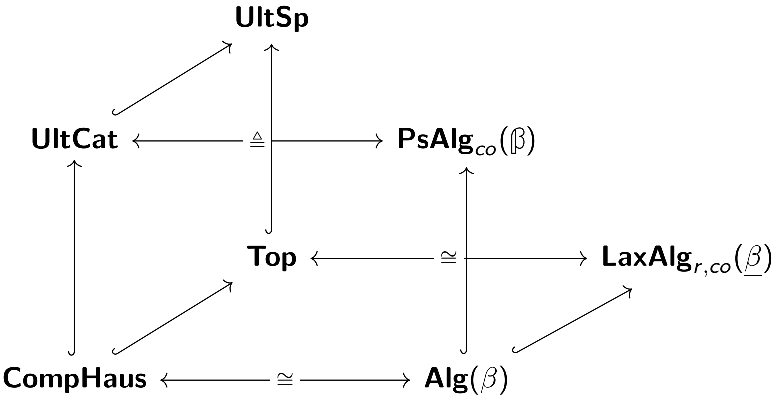

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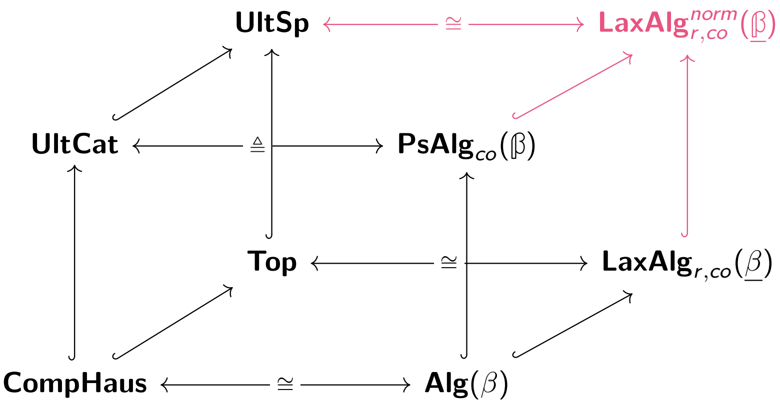

Aristote, T., *Profunctorial algebras*, arXiv:2601.22721

I. Extending pseudomonads

2-dimensional relations

The role of relations, in **CAT**, is played by profunctors:

relation $R \subseteq A \times B$	function $R\colon B \times A \to \mathbf{2}$
profunctor $F\colon C \nrightarrow D$	functor $F\colon D^{\mathsf{op}} \times C \to \mathbf{Set}$

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For our aims, it's simpler to represent profunctors via *cospans* rather than spans.

Fact

Profunctors $C \nrightarrow D$ can be identified with certain cospans $C \to V \leftarrow D$ called two-sided codiscrete cofibrations. Concretely:

- *for $F\colon C \nrightarrow D$, its collage is the cospan $C \xrightarrow{\iota_C} V \xleftarrow{\iota_D} D$ where V is the category having*
  - *as objects, the disjoint union of the objects of C and D;*
  - *as morphisms, the morphisms of C and D together with a morphism $x\colon d \to c$ for each $d \in D$, $c \in C$, and $x \in F(d, c)$,*

  with the evident inclusions ι_C and ι_D;
- *for $C \xrightarrow{q} V \xleftarrow{p} D$, the profunctor $F\colon C \nrightarrow D$ is defined by $F(d, c) = V(p(d), q(c))$.*

Extensions *à la* Barr

The idea is then to study how to extend a pseudofunctor $\mathbf{T}\colon \mathbf{CAT} \to \mathbf{CAT}$ to profunctors by identifying them with their collages,

$$\underline{\mathbf{T}}\left(C \xrightarrow{\iota_C} E \xleftarrow{\iota_D} D \right) \triangleq \mathbf{T}C \overset{(\mathbf{T}\iota_C)_\diamond}{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathbf{T}E \overset{(\mathbf{T}\iota_D)^\diamond}{\relbar\joinrel\mapstochar\joinrel\rightarrow} \mathbf{T}D$$

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- $f^\diamond\colon Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$ is the *cograph* profunctor, $f^\diamond(x, y) = Y(f(x), y)$.

Question

How do we compose profunctors?

Composing relations

First, let's recall how to compose two relations $R \subseteq A \times B$ and $S \subseteq B \times C$:

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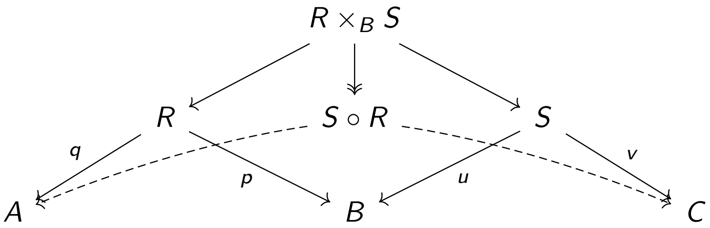
Identifying them with spans, the relation $S \circ R \subseteq A \times C$ arises as the image factorization of the universal functor $R \times_B S \to A \times C$ into the product:



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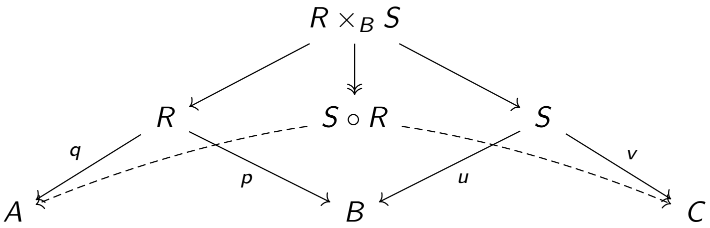

Regularity

In other words, we're using the regularity of **Set** to compose relations, in the sense of a well-behaved factorization system.

Composing profunctors

Similarly, two profunctors $F\colon C \nrightarrow D$ and $G\colon D \nrightarrow E$ can be composed via a coend:

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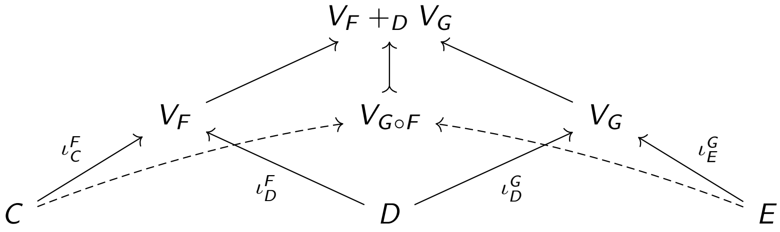
Identifying them with cospans, the composition $G \circ F\colon C \nrightarrow E$ also arises by the image factorization of the universal functor $C + E \to V_F +_D V_G$ out of the coproduct:



The bicategory of profunctors

This composition yields a bicategory **PROF** of categories, profunctors, and natural transformations, which is biequivalent to **TSCodCof**(**CAT**).

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Moreover, graphs and cographs determine identity-on-objects pseudofunctors:

$$(-)_\diamond \colon \mathbf{CAT} \to \mathbf{PROF} \qquad (-)^\diamond \colon \mathbf{CAT}^{\mathrm{op}} \to \mathbf{PROF}^{\mathrm{co}}$$

which are locally fully-faithful and such that $f_\diamond \dashv f^\diamond$ for each functor f.
Such a situation defines a proarrow equipment on **CAT**.

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A familiar example

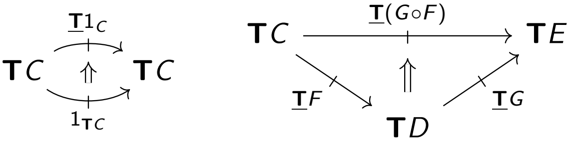
$(-)_\diamond \colon \mathbf{Set} \to \mathbf{Rel}$ is a proarrow equipment.

Extending pseudofunctors I

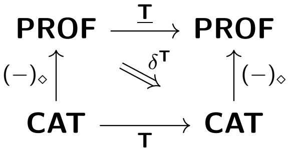
Barr's implicit preservation of surjections here becomes preservation of fully-faithful functors.

Proposition

If $\mathbf{T}$ *preserves fully-faithfulness, the extension* $\underline{\mathbf{T}}\colon \mathbf{PROF} \to \mathbf{PROF}$ *is a lax functor:*

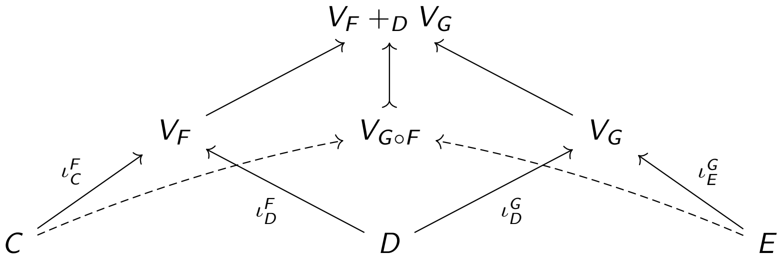


Moreover, $\underline{\mathbf{T}}$ *is equipped with a canonical lax transformation* $\delta^{\mathbf{T}}\colon \underline{\mathbf{T}}(-)_\diamond \Rightarrow (\mathbf{T}-)_\diamond$*:*



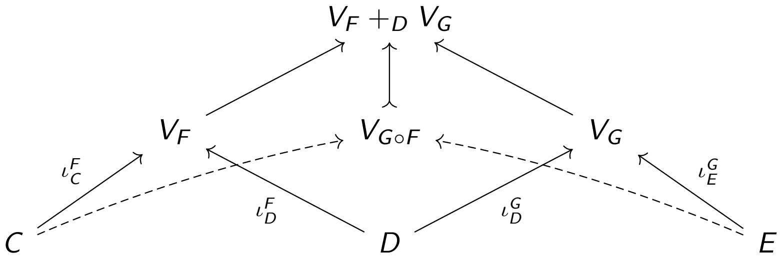
Exact squares

Let's reconsider the composition $G \circ F \colon C \nrightarrow E$:



Exact squares

Let's reconsider the composition $G \circ F \colon C \nrightarrow E$:

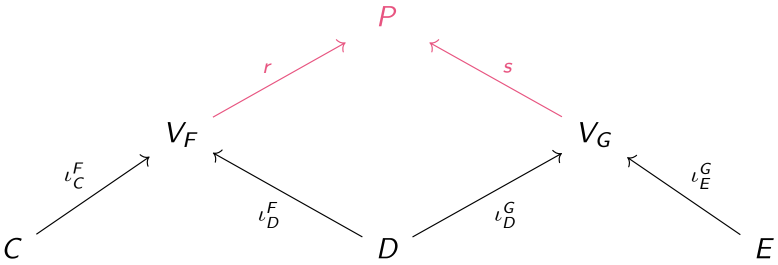


Since $F \cong (\iota_D^F)^\diamond \circ (\iota_C^F)_\diamond$ and $G \cong (\iota_E^G)^\diamond \circ (\iota_D^G)_\diamond$, we always have:

$$G \circ F \cong (\iota_E^G)^\diamond \circ (\iota_D^G)_\diamond \circ (\iota_D^F)^\diamond \circ (\iota_C^F)_\diamond$$

Exact squares

Suppose that we can find a square...



...such that $(\iota^G_D)_\diamond \circ (\iota^F_D)^\diamond \cong s^\diamond \circ r_\diamond$. Then:

$$G \circ F \cong (\iota^G_E)^\diamond \circ (\iota^G_D)_\diamond \circ (\iota^F_D)^\diamond \circ (\iota^F_C)_\diamond$$

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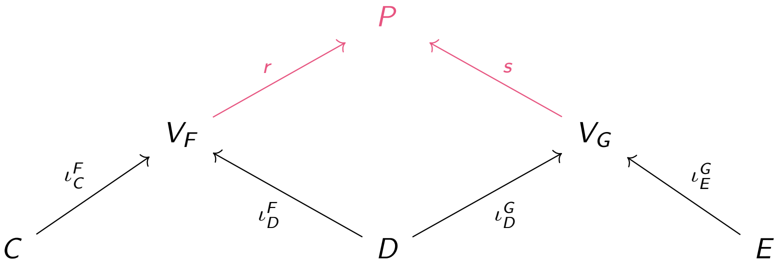

...such that $(\iota_D^G)_\diamond \circ (\iota_D^F)^\diamond \cong s^\diamond \circ r_\diamond$. Then:

$$
\begin{aligned}
G \circ F &\cong (\iota_E^G)^\diamond \circ (\iota_D^G)_\diamond \circ (\iota_D^F)^\diamond \circ (\iota_C^F)_\diamond \\
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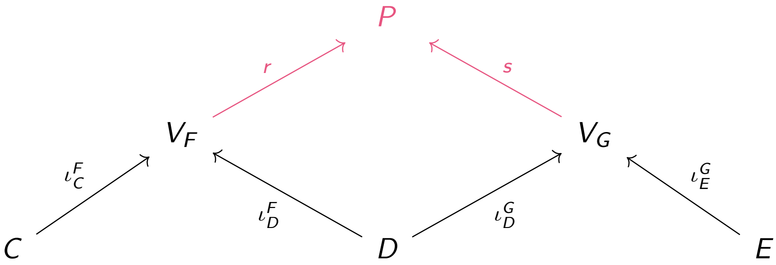
Exact squares

Suppose that we can find a square...



...such that $(\iota_D^G)_\diamond \circ (\iota_D^F)^\diamond \cong s^\diamond \circ r_\diamond$. Then:

$$\begin{aligned} G \circ F &\cong (\iota_E^G)^\diamond \circ (\iota_D^G)_\diamond \circ (\iota_D^F)^\diamond \circ (\iota_C^F)_\diamond \\ &\cong (\iota_E^G)^\diamond \circ s^\diamond \circ r_\diamond \circ (\iota_C^F)_\diamond \\ &\cong (\iota_E^G \circ s)^\diamond \circ (r \circ \iota_C^F)_\diamond \end{aligned}$$

Exact squares

We can give a name to such a square, for an arbitrary equipment $(-)_\diamond \colon \mathbf{K} \to \mathbf{M}$:

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ {\scriptstyle v}\downarrow & \swArrow_{\gamma} & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & D \end{array}$$

Definition

A lax square $\gamma \colon fu \Rightarrow gv$ in $\mathbf{K}$ is exact if the canonical 2-cell below, pasting of the unit of $f_\diamond \dashv f^\diamond$, of $\gamma_\diamond$, and of the counit of $v_\diamond \dashv v^\diamond$, is invertible in $\mathbf{M}$.

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Examples

- With respect to $(-)_\diamond \colon \mathbf{Set} \to \mathbf{Rel}$, exact squares are precisely weak pullbacks.
- With respect to $(-)_\diamond \colon \mathbf{CAT} \to \mathbf{PROF}$, exactness of $\gamma \colon fu \Rightarrow gv$ means that the canonical function below is bijective for all $b \in B$ and $c \in C$:

$$\int^{a \in A} C(u(a), c) \times B(b, v(a)) \to D(f(b), g(c)),$$

Extending pseudofunctors II

Fact

Cocomma squares are exact.

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But then compositions of profunctors can always be computed via exact squares! This gives us the following characterization.

Proposition

The extension $\underline{\mathbf{T}}\colon \mathbf{PROF} \to \mathbf{PROF}$ *is a pseudofunctor if and only if* $\mathbf{T}$ *preserves exact squares, in which case* $\delta^{\mathbf{T}}\colon \underline{\mathbf{T}}(-)_\diamond \Rightarrow (\mathbf{T}-)_\diamond$ *is a pseudonatural isomorphism.*

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Question

Consider the pseudofunctor $\mathrm{Psh}\colon \mathbf{CAT} \to \mathbf{CAT}$ taking categories of presheaves: for $f\colon C \to D$, the functor $\mathrm{Psh}\, f\colon \mathrm{Psh}\, C \to \mathrm{Psh}\, D$ acts by left Kan extension along f^{op}, hence it is fully-faithful if so is f. Does Psh preserve exact squares?

Extending transformations and modifications

Extensions of pseudonatural transformations $\sigma\colon \mathbf{T} \Rightarrow \mathbf{T}'$ and modifications $\mathfrak{m}\colon \sigma \Rrightarrow \sigma'$ can be treated similarly, simply by carrying them along $(-)_\diamond\colon \mathbf{CAT} \to \mathbf{PROF}$.

Proposition

The extension $\underline{\sigma}\colon \underline{\mathbf{T}} \Rightarrow \underline{\mathbf{T}}'$ *defined by* $\underline{\sigma}_C = (\sigma_C)_\diamond$ *is a lax natural transformation, and it is pseudonatural if and only if the naturality squares of* σ *are exact.*

$$\begin{array}{ccc} \mathbf{T}C & \xrightarrow{\sigma_C} & \mathbf{T}'C \\ {\scriptstyle \underline{\mathbf{T}}F}\,\big\downarrow\!\!\!\!+ & \overset{\underline{\sigma}_F}{\swarrow\!\!\!\!\swarrow} & +\!\!\!\!\big\downarrow\,{\scriptstyle \underline{\mathbf{T}}'F} \\ \mathbf{T}D & \xrightarrow[\sigma_D]{} & \mathbf{T}'D \end{array}$$

Proposition

The extension $\underline{\mathfrak{m}}\colon \underline{\sigma} \Rrightarrow \underline{\sigma}'$ *defined by* $\underline{\mathfrak{m}}_C = (\mathfrak{m}_C)_\diamond$ *is a modification.*

Extending pseudomonads

Combining these results, we obtain an extension result for pseudomonads on **CAT**.

Theorem

Every pseudomonad $\langle \mathbf{T}, \eta^{\mathbf{T}}, \mu^{\mathbf{T}} \rangle$ *on* **CAT** *preserving fully-faithfulness extends to a left skew monad* $\langle \underline{\mathbf{T}}, \eta^{\underline{\mathbf{T}}}, \mu^{\underline{\mathbf{T}}} \rangle$ *on* **PROF**, *which is a pseudomonad if and only if:*

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Unit and multiplication

To construct $\eta^{\underline{\mathbf{T}}}$ and $\mu^{\underline{\mathbf{T}}}$ we use two canonical transformations with identity components:

- $\omega^1 \colon 1_{\mathbf{PROF}} \Rightarrow \underline{1_{\mathbf{CAT}}}$, which is always pseudonatural;
- $\omega^{\mathbf{T}} \colon \underline{\mathbf{T}} \circ \underline{\mathbf{T}} \Rightarrow \underline{\mathbf{T} \circ \mathbf{T}}$, which is generally lax natural but pseudonatural if $\mathbf{T}$ preserves exact squares.

Left oplax Kan extensions

We now introduce a class of pseudomonads on **CAT** which extend to skew monads on **PROF**. Fix a **Set**-monad $\langle T, \eta, \mu \rangle$ and compose T with the inclusion **Set** $\hookrightarrow$ **CAT**.

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Definition (T., Wrigley)

If it exists, the left oplax Kan extension of T is the unique 2-functor $\mathbf{L}_T \colon \mathbf{CAT} \to \mathbf{CAT}$ characterized by a pseudonatural isomorphism:

$$\mathbf{Str}\,[\mathbf{CAT}, \mathbf{CAT}]\,(\mathbf{L}_T, \mathbf{S}) \cong \mathbf{Oplax}\,[\mathbf{Set}, \mathbf{CAT}]\,(T, \mathbf{S}|_{\mathbf{Set}})$$

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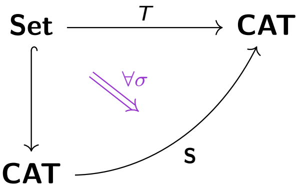

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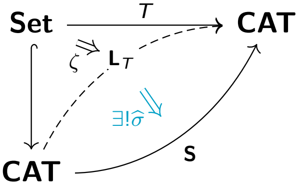

Left oplax Kan extensions

Theorem (T., Wrigley)

$\mathbf{L}_T$ *exists, and it carries the structure of a pseudomonad.*

For a category C, the category $\mathbf{L}_T C$ has:

- as objects, triples (I, y, ν) of a set I, a functor $y \colon I \to C$, and an element $\nu \in TI$;
- as morphisms $(I, y, \nu) \to (I', y', \nu')$, pairs of a function $h \colon I' \to I$ such that $Th(\nu') = \nu$ and a family of arrows $(\alpha_i \colon y_{h(i)} \to y'_i)_{i \in I'}$ in C.

For $f \colon C \to D$, the functor $\mathbf{L}_T f$ acts by postcomposition, i.e. $\mathbf{L}_T(I, y, \nu) = (I, fy, \nu)$, so it clearly preserves fully-faithfulness.

Corollary

$\mathbf{L}_T$ *extends to a left skew monad on* **PROF**.

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3. The map $q \colon X \to \beta Y$ defined by $x \mapsto \beta i_x(\theta_x)$ extends to a map $q^* \colon \beta X \to \beta Y$, so that we can consider $q^*(\nu) \in \beta Y$. Concretely, for $S \subseteq Y$,

$$S \in q^*(\nu) \iff \{ x \in X \mid S \cap Y_x \in \theta_x \} \in \nu$$

Quotient by almost-everywhere equality

β is a quotient of $\mathbf{L}_\beta$

The category βC is precisely the quotient of $\mathbf{L}_\beta C$ given identifying morphisms $(I, y, \nu) \to (I', y', \nu')$ up to equality on a ν'-large subset of I'.

Quotient by almost-everywhere equality

$\mathbb{\beta}$ is a quotient of $\mathbf{L}_\beta$

The category $\mathbb{\beta}C$ is precisely the quotient of $\mathbf{L}_\beta C$ given identifying morphisms $(I, y, \nu) \to (I', y', \nu')$ up to equality on a ν'-large subset of I'.

In fact, many monads of interest yield a notion of *almost-everywhere equality* which we can use to quotient each category $\mathbf{L}_T C$ out, obtaining 'quotient pseudomonads' $\mathbf{Q}_T$:

- for $T = \mathcal{F}$ the filter monad, as for β, we can identify two morphisms $(I, y, \nu \in \mathcal{F}I) \to (I', y', \nu' \in \mathcal{F}I)$ if they agree on a ν'-large subset of I';
- for $T = \mathcal{P}$ the powerset monad, we can identify two morphisms $(I, y, \nu \in \mathcal{P}I) \to (I', y', \nu' \in \mathcal{P}I)$ if they agree pointwise on $\nu' \subseteq I'$;
- for $T = \mathcal{D}$ the distribution monad, we can identify two morphisms $(I, y, \nu \in \mathcal{D}I) \to (I', y', \nu' \in \mathcal{D}I)$ if they agree on the support of ν'.

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- for $T = \mathcal{D}$ the distribution monad, we can identify two morphisms $(I, y, \nu \in \mathcal{D}I) \to (I', y', \nu' \in \mathcal{D}I)$ if they agree on the support of ν'.

Corollary

For $T \in \{\beta, \mathcal{F}, \mathcal{P}, \mathcal{D}\}$, $\mathbf{Q}_T$ extends to a left skew monad on **PROF**.

II. Recovering ultraconvergence spaces

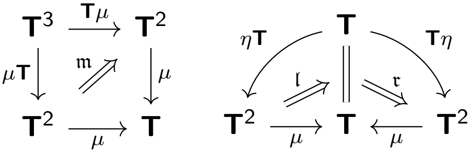
Left skew monads

So now we know that, as β extends to a right skew monad on **Rel**, so β extends to a left skew monad on **PROF**; let us finally see this definition spelled out.

Definition

A left skew monad on a bicategory **K** consists of:

- a lax functor $\mathbf{T}\colon \mathbf{K} \to \mathbf{K}$, with structural 2-cells $\psi_A\colon 1_{\mathbf{T}A} \Rightarrow \mathbf{T}1_A$ and $\psi_{g,f}\colon \mathbf{T}g \circ \mathbf{T}f \Rightarrow \mathbf{T}(g \circ f)$;
- lax natural *unit* $\eta\colon 1 \Rightarrow \mathbf{T}$ and *multiplication* $\mu\colon \mathbf{T}^2 \Rightarrow \mathbf{T}$;
- three modifications



satisfying appropriate coherence conditions.

Ultraconvergence spaces as discrete algebras I

Let us recall two definitions that we saw earlier.

Definition

An ultraconvergence space consists of a discrete category X together with a profunctor $\Xi \colon \beta X \nrightarrow X$ defining *ultra-arrows*. Moreover, X is equipped with:

- for every $x \in X$, an *identity* ultra-arrow $\mathrm{id}_x \colon x \prec_{*:1} x$;
- for every ultra-arrow $r \colon x \prec_{i:\mu} y_i$ and every ultrafamily of ultra-arrows $(s_i \colon y_i \prec_{j:\nu_i} z_{i,j})_{i:\mu}$, a *composite* ultra-arrow $(s_i)_{i:\mu} \cdot r \colon x \prec_{(i,j):\sum_{i:\mu} \nu_i} z_{i,j}$,

satisfying equational axioms which express unitality and associativity of composition.

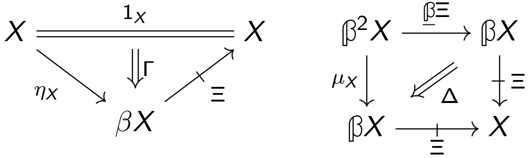
Definition

A continuous map of ultraconvergence spaces is a functor $f \colon X \longrightarrow X'$ together with a family of maps $\Xi(x, (I, y, \nu)) \to \Xi'(f(x), (I, fy, \nu))$, also satisfying equational axioms.

Ultraconvergence spaces as discrete algebras I

Fact

The structure of an ultraconvergence space on a discrete category X coincides with that of a lax $\underline{\beta}$-algebra on X, i.e. a profunctor $\Xi \colon \beta X \to\!\!\!\!\!\!\mapsto X$ together with two natural transformations:



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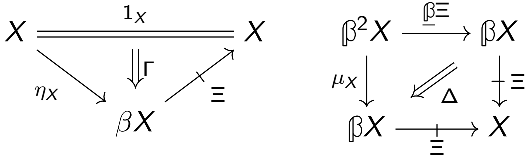
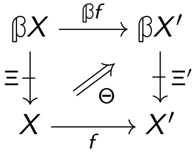


Fact

The structure of a continuous map on a functor $f\colon X \to X'$ coincides with that of a colax morphism of lax $\mathbb{\beta}$-algebras on f, i.e. a natural transformation:



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The notions of lax algebras and their colax morphisms are not well-defined for a left skew monad which is not a pseudomonad.

Second problem

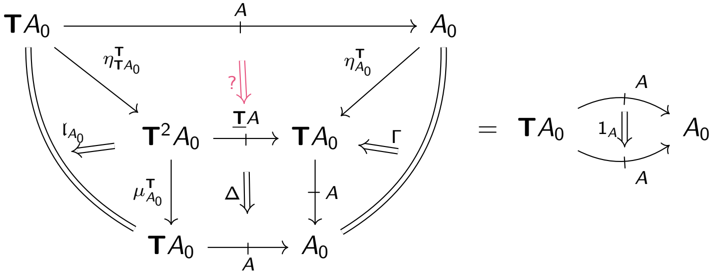
Algebra 2-cells, however defined, trivialize. How do transformations between continuous maps arise?

A coherence problem

Unlike the 1-dimensional case, here we have to consider appropriate coherence conditions which are not evident in this lax setting unless we assume more properties.

A concrete example

For a lax $\underline{\mathbf{T}}$-algebra $A\colon \mathbf{T}A_0 \mathrel{-\!\!\!+\!\!\!\!\rightarrow} A_0$, there should be a 'right unitality' axiom



but, being $\eta^{\underline{\mathbf{T}}}$ *lax* natural, its structural 2-cell goes in the wrong way.

- For ultraconvergence spaces, this should read as $(r)_{*:1}\cdot \mathrm{id}_x = r$ for all $r\colon x \relbar\!\!\prec_{i:\mu} y_i$.

If we allow for lax algebras...

Definition

The skew extension $\langle \underline{\mathbf{T}}, \eta^{\underline{\mathbf{T}}}, \mu^{\underline{\mathbf{T}}} \rangle$ of a pseudomonad $\langle \mathbf{T}, \eta^{\mathbf{T}}, \mu^{\mathbf{T}} \rangle$ allows for lax algebras if:

1. the naturality squares of $\eta^{\mathbf{T}}$ and $\mu^{\mathbf{T}}$ are exact;
2. the transformations $\delta^{\mathbf{T}} \colon \underline{\mathbf{T}}(-)_\diamond \Rightarrow (\mathbf{T}-)_\diamond$ and $\omega^{\mathbf{T}} \colon \underline{\mathbf{T}} \circ \underline{\mathbf{T}} \Rightarrow \underline{\mathbf{T} \circ \mathbf{T}}$ are pseudonatural;
3. there exists a natural family of natural transformations $\varphi_{G,f_\diamond} \colon \underline{\mathbf{T}}(G \circ f_\diamond) \Rightarrow \underline{\mathbf{T}}G \circ \underline{\mathbf{T}}f_\diamond$, for any functor $f \colon A \to B$ and any profunctor $G \colon B \nrightarrow C$, such that for each pair of functors $f \colon A \to B, f' \colon B \to C$ and each profunctor $G \colon C \nrightarrow D$:

   3.1 the diagram below commutes;

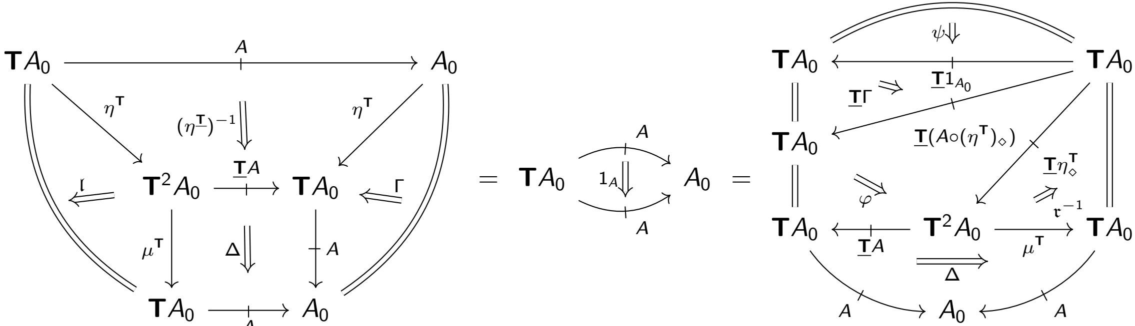
$$\begin{array}{ccccc}
\underline{\mathbf{T}}(G \circ (f'f)_\diamond) & \xrightarrow{\varphi_{G,(f'f)_\diamond}} & \underline{\mathbf{T}}G \circ \underline{\mathbf{T}}(f'f)_\diamond & \xrightarrow{\cong} & \underline{\mathbf{T}}G \circ \underline{\mathbf{T}}(f'_\diamond \circ f_\diamond) \\
{\scriptstyle\cong}\downarrow & & & & \downarrow{\scriptstyle \underline{\mathbf{T}}G * \varphi_{f'_\diamond, f_\diamond}} \\
\underline{\mathbf{T}}(G \circ f'_\diamond \circ f_\diamond) & \xrightarrow[\varphi_{G \circ f'_\diamond, f_\diamond}]{} & \underline{\mathbf{T}}(G \circ f'_\diamond) \circ \underline{\mathbf{T}}f_\diamond & \xrightarrow[\varphi_{G, f'_\diamond} * \underline{\mathbf{T}}f_\diamond]{} & \underline{\mathbf{T}}G \circ \underline{\mathbf{T}}f'_\diamond \circ \underline{\mathbf{T}}f_\diamond
\end{array}$$

   3.2 $\varphi_{f'_\diamond, f_\diamond} = \psi^{-1}_{f'_\diamond, f_\diamond}$, where $\psi_{f'_\diamond, f_\diamond} \colon \underline{\mathbf{T}}f'_\diamond \circ \underline{\mathbf{T}}f_\diamond \Rightarrow \underline{\mathbf{T}}(f'_\diamond \circ f_\diamond)$ is the lax associator for $\underline{\mathbf{T}}$.

...then lax algebras are allowed

In essence, a left skew extension $\langle \underline{\mathbf{T}}, \eta^{\underline{\mathbf{T}}}, \mu^{\underline{\mathbf{T}}} \rangle$ allowing for lax algebras is *really* close to being a pseudomonad, enough so that we can define the 2-category $\mathbf{LaxAlg}_{r,co}(\underline{\mathbf{T}})$ of:

- lax $\underline{\mathbf{T}}$-algebras;



TA_0 A A_0
η^T (η^T)^{-1} η^T
l T^2A_0 TA TA_0 Γ
μ^T Δ A
TA_0 A A_0
=
TA_0 A 1_A A A_0
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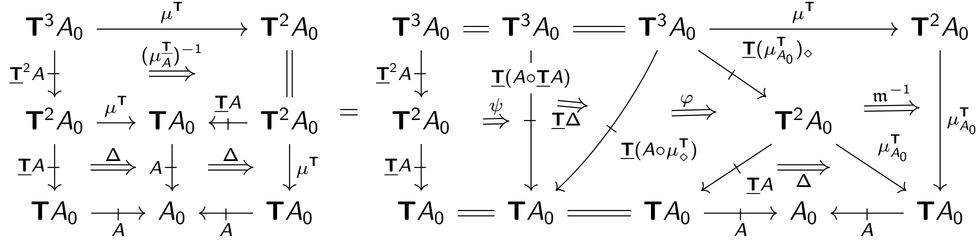
TA_0 TA T^2A_0 μ^T TA_0
r^{-1}
Δ
A A_0 A



T^3A_0 μ^T T^2A_0
T^2A (μ^T_A)^{-1}
T^2A_0 μ^T TA_0 TA T^2A_0
TA Δ A Δ μ^T
TA_0 A A_0 A TA_0
=
T^3A_0 T^3A_0 T^3A_0 μ^T T^2A_0
T^2A T(A∘TA) T(μ^T_{A_0})_⋄
T^2A_0 ψ TΔ φ T^2A_0 m^{-1} μ^T_{A_0}
TA T(A∘μ^T_⋄) μ^T_{A_0}
TA Δ
TA_0 TA_0 TA_0 A A_0 A TA_0

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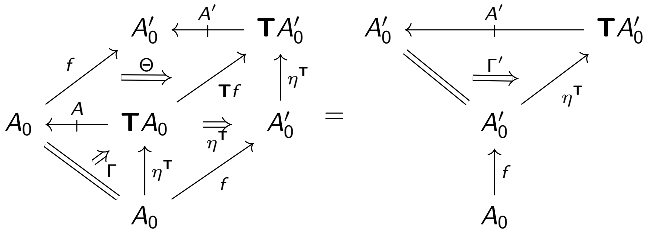
A'
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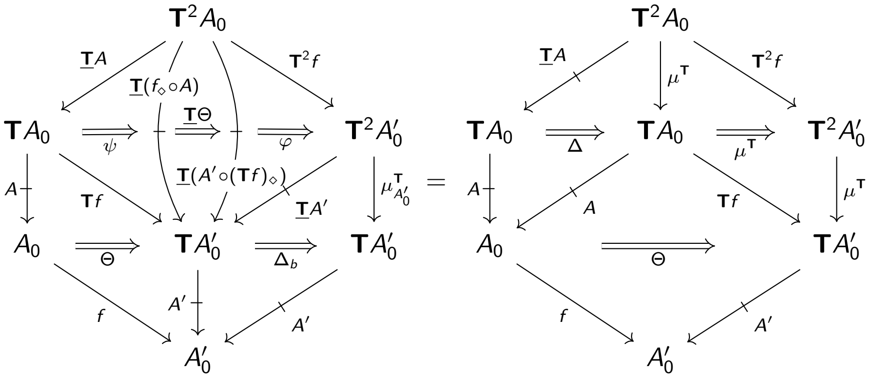
TA
T²f
T(f◇∘A)
TΘ
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T(A'∘(Tf)◇)
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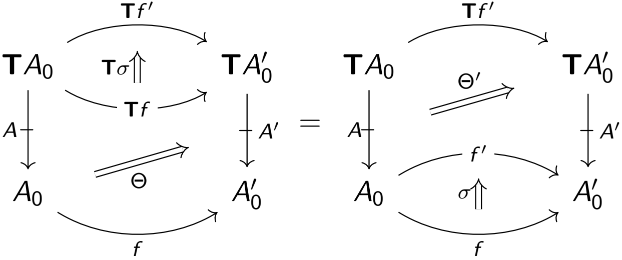

Tf′
TA0
Tσ⇑
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Proposition

For any **Set**-*monad* T, *the pseudomonad* $\mathbf{L}_T$ *allows for lax algebras.*
For $T \in \{ \beta, \mathcal{F}, \mathcal{P}, \mathcal{D} \}$, *the same holds for the pseudomonad* $\mathbf{Q}_T$*.*

Ultraconvergence spaces as discrete algebras II

Focusing on the case $T = \beta$, we then have that the axioms of an ultraconvergence space correspond to those of a lax $\underline{\mathbb{\beta}}$-algebra, and similarly for continuous maps.

Lemma

There is an isomorphism of categories between:

1. *ultraconvergence spaces and continuous maps;*
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Crucially, $\mathbf{LaxAlg}^{disc}_{r,co}(\underline{\mathbb{\beta}})$ is locally discrete, since its 2-cells are based on natural transformations between functors between discrete categories.

Categories of points

To address this, we draw inspiration from the theory of generalized multicategories.

Definition

For a lax $\underline{\beta}$-algebra $A\colon \beta A_0 \nrightarrow A_0$, we define its category of points $(\mathrm{pt}\,A)_0$ as the category having:

- as objects, the objects of A_0;
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A extends to a profunctor $\mathrm{pt}\, A \colon \beta(\mathrm{pt}\, A)_0 \nrightarrow (\mathrm{pt}\, A)_0$, also carrying the structure of a lax $\underline{\beta}$-algebra. The operation $A \mapsto \mathrm{pt}\, A$ is a closure operator on $\mathbf{LaxAlg}_{r,co}(\underline{\beta})$:

- $\mathrm{pt}(\mathrm{pt}\, A) \cong \mathrm{pt}\, A$;
- there is an identity-on-objects functor $\gamma_A \colon A_0 \to (\mathrm{pt}\, A)_0$ carrying the structure of a colax morphism $A \to \mathrm{pt}\, A$;

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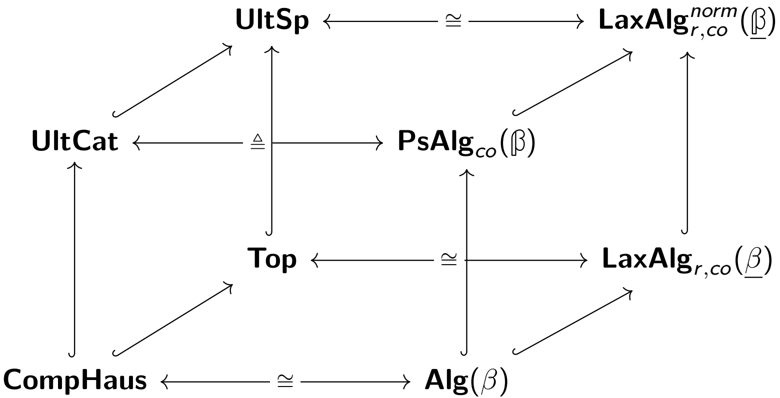
For 2-cells, instead, it makes all the difference: in the normalized description, algebra 2-cells recover the transformations between continuous maps.

Theorem

There is an isomorphism of 2-categories $\mathbf{UltSp} \cong \mathbf{LaxAlg}^{norm}_{r,co}(\underline{\beta})$.

Wrap-up

It is immediate to see that $\mathbb{\beta}$-algebras embed into normalized lax $\underline{\mathbb{\beta}}$-algebras, so that we conclude with the desired picture.



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Thank you!

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