Conceptual completeness for geometric logic via ultraconvergence spaces

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Introduction and motivation

Ultracategories

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For first-order logic, the role of Lindenbaum-Tarski algebras is played by classifying toposes: theories with equivalent classifying toposes have essentially the same models.

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For first-order logic, the role of Lindenbaum-Tarski algebras is played by classifying toposes: theories with equivalent classifying toposes have essentially the same models.

Theorem (Makkai; Lurie)

For a coherent theory $\mathbb{T}$, the category $\mathbf{Mod}(\mathbb{T})$ is an ultracategory, and $\mathbf{UltCat}(\mathbf{Mod}(\mathbb{T}), \mathbf{Set})$ is the classifying topos of $\mathbb{T}$.

Ultracategories

At its core, an ultracategory is a category C equipped with abstract ultraproducts, i.e. a functorial choice of a C-object $\prod_{i:\nu} c_i$ for each ultrafamily $(c_i)_{i\to\nu}$ in C, the datum of

- a set I,
- an I-indexed family of C-objects $(c_i)_{i\in I}$, and
- an ultrafilter $\nu \in \beta I$,

behaving as ultraproducts of sets.

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With appropriate morphisms, the category $\mathbb{\beta} C$ of ultrafamilies in C gives rise to a pseudomonad $\mathbb{\beta}\colon \mathbf{CAT} \to \mathbf{CAT}$, dubbed the ultracompletion pseudomonad by Rosolini.

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With appropriate morphisms, the category βC of ultrafamilies in C gives rise to a pseudomonad $\beta\colon \mathbf{CAT} \to \mathbf{CAT}$, dubbed the ultracompletion pseudomonad by Rosolini.

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UltCat is the 2-category of pseudo-β-algebras, colax morphisms, and algebra 2-cells.

Ultracategories categorify compact Hausdorff spaces

CompHaus $\hookrightarrow$ **UltCat** as the small and discrete ultracategories.

Ultracategories and coherent toposes

Identifying coherent theories with coherent toposes, and restricting to *bounded* ultracategories, i.e. those ultracategories C such that $\mathbf{UltCat}(C, \mathbf{Set})$ is a topos, we have:

$$\begin{tikzcd} \mathbf{Topos}_{coh} \arrow[rr, phantom, "\perp"] \arrow[rr, hook, bend right, "\mathrm{pt}(-)"'] & & \mathbf{UltCat}_{bnd} \arrow[ll, bend right, "\mathbf{UltCat}(-{,}\mathbf{Set})"'] \end{tikzcd}$$

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This result crucially rests on Łoś's theorem: for a coherent theory, ultraproducts of models are themselves models. In other words, the ultraproduct functors

$$\mathbf{Set}^I \xrightarrow{\prod_{i:\nu}(-)} \mathbf{Set}$$

are coherent, i.e. they preserve finite limits, regular epis, and finite unions of subobjects.

What about geometric logic?

We will now consider geometric logic: a theory is geometric if its axioms are of the form $\forall \vec{x}(\varphi(\vec{x}) \to \psi(\vec{x}))$ where φ, ψ are built only using finitary $\wedge$, infinitary $\vee$, and $\exists$.

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A necessary restriction

Having such a result, for a geometric theory $\mathbb{T}$, entails its completeness with respect to its **Set**-models. Categorically, this corresponds to restricting to toposes with enough points, a condition analogous to *spatiality* for locales.

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Ultraconvergence spaces

Topology and convergence of ultrafilters

The key intuition to address this question comes from Barr's *Relational algebras*.

Theorem (Barr)

The ultrafilter monad $\beta\colon \mathbf{Set} \to \mathbf{Set}$ *extends to a skew monad* $\underline{\beta}$ *on* $\mathbf{Rel}$, *and* $\mathbf{Top} \cong \mathbf{LaxAlg}_{r,co}(\underline{\beta})$.

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Concretely, this means that a topology on a set X can be equivalently specified by a relation $\xi \colon \beta X \nrightarrow X$ satisfying

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X \arrow[rr, equal] \arrow[dr, "\eta_X"'] & \arrow[d, phantom, "\rotatebox{90}{$\supseteq$}"] & X \\
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which we can think of as a notion of convergence. In particular:

- $\xi(x,\nu) \iff \forall U \subseteq X$ open, if $x \in U$ then $U \in \nu$;
- U open $\iff \forall x \in U, \nu \in \beta X$, if $\xi(x,\nu)$ then $U \in \nu$.

Topology and convergence of ultrafilters

Remark

We recover Manes' theorem, **CompHaus** $\cong$ **Alg**(β), restricting to those topological spaces X whose convergence relation ξ is functional. Indeed:

- X is compact $\iff$ for every $\nu \in \beta X$ there exists *some* $x \in X$ such that $\xi(x, \nu)$;
- X is Hausdorff $\iff$ for every $\nu \in \beta X$ there exists *at most one* $x \in X$ such that $\xi(x, \nu)$.

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Notation

For $f \colon I \to X$ and $\nu \in \beta I$, we write $x \rightsquigarrow \lim_{i \to \nu} f(i)$ in case $\xi(x, \beta f(\nu))$ holds.

Ultraconvergence spaces

Inspired by Barr's extension of Manes' theorem, we thus generalize ultracategories by replacing a $\mathbb{\beta}$-algebra functor with a *profunctor*, and categorifying the description of topological spaces in terms of convergence.

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- for every $x \in X$, an *identity* ultra-arrow $\mathrm{id}_x\colon x \rightsquigarrow \lim_{*\to 1} x$;
- for every ultra-arrow $r\colon x \rightsquigarrow \lim_{i\to\mu} y_i$ and every ultrafamily of ultra-arrows $(s_i\colon y_i \rightsquigarrow \lim_{j\to\nu_i} z_{i,j})_{i\to\mu}$, a *composite* ultra-arrow $(s_i)_{i\to\mu}\cdot r\colon x \rightsquigarrow \lim_{(i,j)\to\sum_{i\to\mu}\nu_i} z_{i,j}$,

satisfying unitality and associativity axioms.

Continuous maps

Similarly, we can categorify the notion of continuity to this **Set**-valued convergence relation, which now becomes *structure* rather than *property*.

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Definition

A continuous map of ultraconvergence spaces is a functor $f\colon X \longrightarrow X'$ together with a family of functions

$$\begin{aligned} \Xi(x,(y_i)_{i\to\nu}) &\longrightarrow \Xi'(f(x),(f(y_i))_{i\to\nu}) \\ r\colon x \rightsquigarrow \lim_{i\to\nu} y_i &\longmapsto f(r)\colon f(x) \rightsquigarrow \lim_{i\to\nu} f(y_i) \end{aligned}$$

satisfying some equational axioms.

With appropriate 2-cells, ultraconvergence spaces define a 2-category **UltSp**.

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- Every topological space is a **2**-valued ultraconvergence space with the usual convergence relation.

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The main theorem

Ultraconvergence spaces allow us to obtain a reconstruction theorem for toposes with enough points.

Theorem (Saadia; Hamad; van Gool, Marquès, T.)

If $\mathcal{E}$ is a topos with enough points, then $\mathcal{E} \simeq \mathbf{UltSp}(\mathrm{pt}(\mathcal{E}), \mathbf{Set})$.

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In other words, restricting to bounded ultracategories and ultraconvergence spaces:

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Étale spaces

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Étale maps over B form a category $\mathrm{Et}(B)$, equivalent to $\mathbf{UltSp}(B, \mathbf{Set})$.

The ultraconvergence space of points of a topos

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- For every object $\varphi \in \mathcal{E}$, we can define an étale space $\pi_\varphi \colon [\![\varphi]\!] \longrightarrow X$ where:
  - the fiber of π_φ at $x \in X$ is given by $x(\varphi)$;
  - an ultra-arrow $(x, v) \rightsquigarrow \lim_{i \to \nu}(y_i, w_i)$ in $[\![\varphi]\!]$ is given by an ultra-arrow $r \colon x \rightsquigarrow \lim_{i \to \nu} y_i$ in X such that $r_\varphi(v) = (w_i)_{i \to \nu}$.

  This assignment defines the *evaluation functor* $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$.

Reconstruction theorem

Theorem

If X is a separating set of points of $\mathcal{E}$, then $[\![-]\!]\colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is an equivalence.

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Theorem

Let $\mathbb{T}$ be a geometric theory which is complete with respect to its **Set***-models. Then,* $\mathrm{Mod}(\mathbb{T})$ *is an ultraconvergence space by setting ultra-arrows $M \rightsquigarrow \lim_{i \to \nu} N_i$ to be structure morphisms $M \to \prod_{i:\nu} N_i$, and* $\mathrm{Et}(\mathrm{Mod}(\mathbb{T}))$ *is the classifying topos of $\mathbb{T}$.*

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The localic/propositional case

In particular, if a localic topos $\mathcal{E}$ has enough points, i.e. $\mathcal{E} \simeq \mathrm{Sh}(\mathcal{O}(X))$ for some topological space X, then $\mathcal{E} \simeq \mathrm{Et}(X)$.

Proof sketch

Our proof is substantially different from both Saadia's and Hamad's, who make use of Butz-Moerdijk's representation theorem for toposes with enough points. Instead, we proceed similarly to Makkai's original work, in two main steps.

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1. $[\![-]\!] \colon \mathcal{E} \longrightarrow \mathrm{Et}(X)$ is full on subobjects: every subobject of $\pi_\varphi \colon [\![\varphi]\!] \longrightarrow X$ in $\mathrm{Et}(X)$ is the restriction of π_φ to $[\![\psi]\!] \subseteq [\![\varphi]\!]$ for some subobject $\psi \rightarrowtail \varphi$ in $\mathcal{E}$.

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2. $[\![-]\!]\colon \mathcal{E} \longrightarrow \mathsf{Et}(X)$ is covering: every étale space $p\colon Y \longrightarrow X$ is covered by an epimorphism $\alpha\colon \pi_\varphi \twoheadrightarrow p$ in $\mathsf{Et}(X)$ for some object $\varphi \in \mathcal{E}$.

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Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $[\![-]\!]$.

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Two points of view

Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $[\![-]\!]$. However, we can also interpret (1) as stating that $[\![-]\!]$ defines a hyperconnected geometric morphism, and (2) as stating that it defines a localic geometric morphism.

Ultraconvergence spaces as algebras

In joint work with Quentin Aristote, we pushed the inspiration from Barr's theorem even further.

Theorem (Aristote, T.)

The ultracompletion pseudomonad $\mathbb{\beta}\colon \mathbf{CAT} \longrightarrow \mathbf{CAT}$ *extends to a skew monad* $\underline{\mathbb{\beta}}$ *on* $\mathbf{PROF}$, *and* $\mathbf{UltSp} \cong \mathbf{LaxAlg}^{norm}_{r,co}(\underline{\mathbb{\beta}})$.

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This means that an ultraconvergence structure on a discrete category X can be equivalently specified by a profunctor $\Xi \colon \beta X \nrightarrow X$ and two transformations

$$
\begin{tikzcd}
X \arrow[rr, equal] \arrow[dr, "\eta_X"'] & {} \arrow[d, phantom, "\Downarrow"] & X \\
& \beta X \arrow[ur, "\Xi"', "|" marking] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\beta^2 X \arrow[r, "\underline{\beta}\Xi", "|" marking] \arrow[d, "\mu_X"'] \arrow[dr, phantom, "\swarrow"] & \beta X \arrow[d, "\Xi", "|" marking] \\
\beta X \arrow[r, "\Xi"', "|" marking] & X
\end{tikzcd}
$$

satisfying usual coherence axioms.

$$
\begin{tikzcd}
 & \mathbf{UltSp} \arrow[rr, leftrightarrow, "\cong" description] & & \mathbf{LaxAlg}^{norm}_{r,co}(\underline{\mathbb{\beta}}) \\
\mathbf{UltCat} \arrow[ur, hook] \arrow[rr, leftrightarrow, "\triangleq" description] & & \mathbf{PsAlg}_{co}(\mathbb{\beta}) \arrow[ur, hook] & \\
 & \mathbf{Top} \arrow[uu, hook] \arrow[rr, leftrightarrow, "\cong" description] & & \mathbf{LaxAlg}_{r,co}(\underline{\beta}) \arrow[uu, hook] \\
\mathbf{CompHaus} \arrow[uu, hook] \arrow[ur, hook] \arrow[rr, leftrightarrow, "\cong" description] & & \mathbf{Alg}(\beta) \arrow[uu, hook] \arrow[ur, hook] &
\end{tikzcd}
$$

Future work

- We can repeat the above story filters and reduced products in place of ultrafilters and ultraproducts: this yields a notion of convergence spaces, categorifying the relational algebras for the filter monad. How do these relate with ultraconvergence spaces? Can we obtain a more constructive version of the reconstruction theorem?

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Thank you!

E. G. Manes (1969). "A triple theoretic construction of compact algebras". In: *Seminar on Triples and Categorical Homology Theory*. Ed. by B. Eckmann. Vol. 80. Lecture notes in Mathematics. Berlin, Heidelberg: Springer, pp. 91–118.

M. Barr (1970). "Relational algebras". In: *Reports of the Midwest Category Seminar IV*. Ed. by S. MacLane et al. Berlin, Heidelberg: Springer, pp. 39–55.

M. Makkai (1987). "Stone duality for first order logic". In: *Advances in Mathematics* 65.2, pp. 97–170.

J. Lurie (2018). *Ultracategories*. Available online.

G. Rosolini (2024). *Ultracompletions*. Talk at CT2024.

G. Saadia (2025). *Extending conceptual completeness via virtual ultracategories*. arXiv: 2506.23935 [math.CT].

A. Hamad (2025). *Generalised ultracategories and conceptual completeness of geometric logic*. arXiv: 2507.07922 [math.CT].

Q. Aristote and U. Tarantino (2026). *Profunctorial algebras*. arXiv: 2601.22721 [math.CT].

S. van Gool, J. Marquès, and U. Tarantino (2026). "Toposes with enough points as categories of étale spaces". In: *Journal of Pure and Applied Algebra* 230.6, p. 108269.