

Conceptual completeness for geometric logic via ultraconvergence spaces

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Introduction and motivation

Ultracategories

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For first-order logic, the role of Lindenbaum-Tarski algebras is played by **classifying toposes**: theories with equivalent classifying toposes have essentially the same models.

Theorem (Makkai; Lurie)

*For a coherent theory $\mathbb{T}$, the category **Mod**($\mathbb{T}$) is an ultracategory, and **UltCat**(**Mod**($\mathbb{T}$), **Set**) is the classifying topos of $\mathbb{T}$.*

Ultracategories

At its core, an ultracategory is a category C equipped with abstract **ultraproducts**, i.e. a functorial choice of a C -object $\prod_{i:\nu} c_i$ for each **ultrafamily** $(c_i)_{i \rightarrow \nu}$ in C , the datum of

- ▶ a set I ,
- ▶ an I -indexed family of C -objects $(c_i)_{i \in I}$, and
- ▶ an ultrafilter $\nu \in \beta I$,

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Ultracategories categorify compact Hausdorff spaces

CompHaus $\hookrightarrow$ **UltCat** as the small and discrete ultracategories.

Ultracategories and coherent toposes

Identifying coherent theories with coherent toposes, and restricting to *bounded* ultracategories, i.e. those ultracategories C such that $\mathbf{UltCat}(C, \mathbf{Set})$ is a topos, we have:

$$\begin{array}{ccc} & \mathbf{UltCat}(-, \mathbf{Set}) & \\ \swarrow & & \searrow \\ \mathbf{Topos}_{coh} & \perp & \mathbf{UltCat}_{bnd} \\ \nwarrow & & \nearrow \\ & \mathbf{pt}(-) & \end{array}$$

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This result crucially rests on **Łoś's theorem**: for a coherent theory, ultraproducts of models are themselves models. In other words, the ultraproduct functors

$$\mathbf{Set}' \xrightarrow{\Pi_{i:\nu}(-)} \mathbf{Set}$$

are **coherent**, i.e. they preserve finite limits, regular epis, and finite unions of subobjects.

What about geometric logic?

We will now consider **geometric logic**: a theory is geometric if its axioms are of the form $\forall \vec{x}(\varphi(\vec{x}) \rightarrow \psi(\vec{x}))$ where φ, ψ are built only using finitary $\wedge$, **infinitary** $\vee$, and $\exists$. For a geometric theory, Łoś's theorem fails: the ultraproduct functors are not **geometric**, as they don't preserve arbitrary unions of subobjects.

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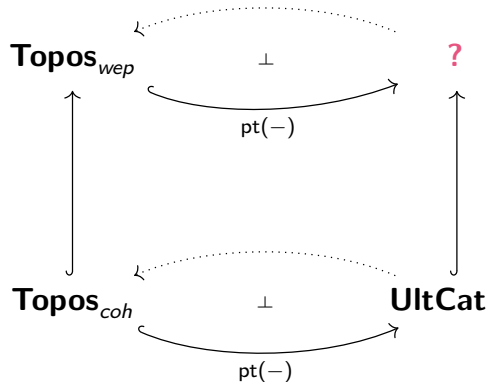
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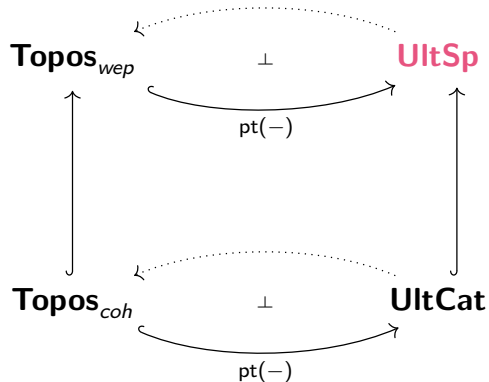
A necessary restriction

Having such a result, for a geometric theory $\mathbb{T}$, entails its completeness with respect to its **Set**-models. Categorically, this corresponds to restricting to toposes **with enough points**, a condition analogous to *spatiality* for locales.

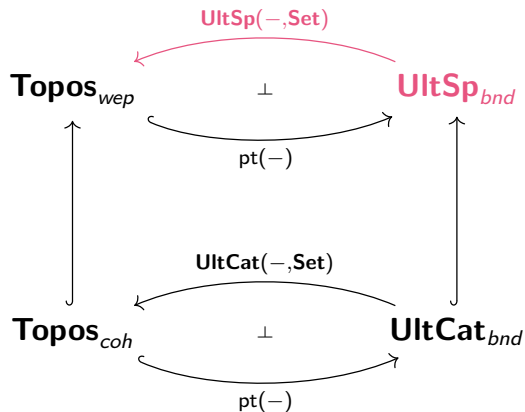
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Ultraconvergence spaces

Topology and convergence of ultrafilters

The key intuition to address this question comes from Barr's *Relational algebras*.

Theorem (Barr)

The ultrafilter monad $\beta: \mathbf{Set} \rightarrow \mathbf{Set}$ extends to a skew monad $\underline{\beta}$ on $\mathbf{Rel}$, and $\mathbf{Top} \cong \mathbf{LaxAlg}_{r,co}(\underline{\beta})$.

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Concretely, this means that a topology on a set X can be equivalently specified by a relation $\xi: \beta X \rightarrowtail X$ satisfying

$$\begin{array}{ccc} X & \xlongequal{\quad} & X \\ \eta_X \searrow & \cap & \nearrow \xi \\ & \beta X & \end{array} \qquad \begin{array}{ccccc} \beta^2 X & \xrightarrow[\top]{\underline{\beta}\xi} & \beta X & & \\ \mu_X \downarrow & \wr & \downarrow \xi & & \\ \beta X & \xrightarrow[\xi]{\top} & X & & \end{array}$$

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which we can think of as a notion of **convergence**. In particular:

- ▶ $\xi(x, \nu) \iff \forall U \subseteq X \text{ open, if } x \in U \text{ then } U \in \nu$;
- ▶ $U \text{ open} \iff \forall x \in U, \nu \in \beta X, \text{ if } \xi(x, \nu) \text{ then } U \in \nu$.

Topology and convergence of ultrafilters

Remark

We recover Manes' theorem, $\mathbf{CompHaus} \cong \mathbf{Alg}(\beta)$, restricting to those topological spaces X whose convergence relation ξ is functional. Indeed:

- ▶ X is compact $\iff$ for every $\nu \in \beta X$ there exists *some* $x \in X$ such that $\xi(x, \nu)$;
- ▶ X is Hausdorff $\iff$ for every $\nu \in \beta X$ there exists *at most one* $x \in X$ such that $\xi(x, \nu)$.

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Notation

For $f: I \rightarrow X$ and $\nu \in \beta I$, we write $x \rightsquigarrow \lim_{i \rightarrow \nu} f(i)$ in case $\xi(x, \beta f(\nu))$ holds.

Ultraconvergence spaces

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- ▶ for every $x \in X$, an *identity* ultra-arrow $\text{id}_x: x \rightsquigarrow \lim_{* \rightarrow 1} x$;
- ▶ for every ultra-arrow $r: x \rightsquigarrow \lim_{i \rightarrow \mu} y_i$ and every ultrafamily of ultra-arrows $(s_i: y_i \rightsquigarrow \lim_{j \rightarrow \nu_i} z_{i,j})_{i \rightarrow \mu}$, a *composite* ultra-arrow $(s_i)_{i \rightarrow \mu} \cdot r: x \rightsquigarrow \lim_{(i,j) \rightarrow \sum_{i \rightarrow \mu} \nu_i} z_{i,j}$,

satisfying unitality and associativity axioms.

Continuous maps

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Definition

A **continuous map** of ultraconvergence spaces is a functor $f: X \longrightarrow X'$ together with a family of functions

$$\begin{aligned}\Xi(x, (y_i)_{i \rightarrow \nu}) &\longrightarrow \Xi'(f(x), (f(y_i))_{i \rightarrow \nu}) \\ r: x \rightsquigarrow \lim_{i \rightarrow \nu} y_i &\longmapsto f(r): f(x) \rightsquigarrow \lim_{i \rightarrow \nu} f(y_i)\end{aligned}$$

satisfying some equational axioms.

With appropriate 2-cells, ultraconvergence spaces define a 2-category **UltSp**.

Examples

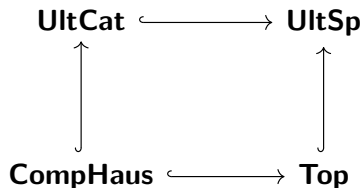
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- ▶ Every ultracategory C , defined by a functor $\Phi: \mathbb{B}C \longrightarrow C$, is an ultraconvergence space by setting ultra-arrows $c \rightsquigarrow \lim_{i \rightarrow \nu} d_i$ to be arrows $c \rightarrow \Phi((d_i)_{i \rightarrow \nu})$ in C .

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The main theorem

Ultraconvergence spaces allow us to obtain a reconstruction theorem for toposes with enough points.

Theorem (Saadia; Hamad; van Gool, Marquès, T.)

If $\mathcal{E}$ is a topos with enough points, then $\mathcal{E} \simeq \mathbf{UltSp}(\mathbf{pt}(\mathcal{E}), \mathbf{Set})$.

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In other words, restricting to bounded ultracategories and ultraconvergence spaces:

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Étale maps over B form a category $\text{Et}(B)$, equivalent to $\mathbf{UltSp}(B, \mathbf{Set})$.

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- ▶ For every object $\varphi \in \mathcal{E}$, we can define an étale space $\pi_\varphi: \llbracket \varphi \rrbracket \longrightarrow X$ where:
 - ▶ the fiber of π_φ at $x \in X$ is given by $x(\varphi)$;
 - ▶ an ultra-arrow $(x, \nu) \rightsquigarrow \lim_{i \rightarrow \nu} (y_i, w_i)$ in $\llbracket \varphi \rrbracket$ is given by an ultra-arrow $r: x \rightsquigarrow \lim_{i \rightarrow \nu} y_i$ in X such that $r_\varphi(\nu) = (w_i)_{i \rightarrow \nu}$.

This assignment defines the *evaluation functor* $\llbracket - \rrbracket: \mathcal{E} \longrightarrow \text{Et}(X)$.

Reconstruction theorem

Theorem

If X is a separating set of points of $\mathcal{E}$, then $\llbracket - \rrbracket: \mathcal{E} \longrightarrow \text{Et}(X)$ is an equivalence.

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Although we need X to be small to prove the above result, it follows easily that $\mathcal{E} \simeq \text{Et}(\text{pt}(\mathcal{E}))$. In logical terms, this reads as the desired reconstruction result.

Theorem

*Let $\mathbb{T}$ be a geometric theory which is complete with respect to its **Set**-models. Then, $\text{Mod}(\mathbb{T})$ is an ultraconvergence space by setting ultra-arrows $M \rightsquigarrow \lim_{i \rightarrow \nu} N_i$ to be structure morphisms $M \rightarrow \prod_{i:\nu} N_i$, and $\text{Et}(\text{Mod}(\mathbb{T}))$ is the classifying topos of $\mathbb{T}$.*

Reconstruction theorem

Theorem

If X is a separating set of points of $\mathcal{E}$, then $\llbracket - \rrbracket: \mathcal{E} \longrightarrow \text{Et}(X)$ is an equivalence.

Although we need X to be small to prove the above result, it follows easily that $\mathcal{E} \simeq \text{Et}(\text{pt}(\mathcal{E}))$. In logical terms, this reads as the desired reconstruction result.

Theorem

*Let $\mathbb{T}$ be a geometric theory which is complete with respect to its **Set**-models. Then, $\text{Mod}(\mathbb{T})$ is an ultraconvergence space by setting ultra-arrows $M \rightsquigarrow \lim_{i \rightarrow \nu} N_i$ to be structure morphisms $M \rightarrow \prod_{i:\nu} N_i$, and $\text{Et}(\text{Mod}(\mathbb{T}))$ is the classifying topos of $\mathbb{T}$.*

The localic/propositional case

In particular, if a localic topos $\mathcal{E}$ has enough points, i.e. $\mathcal{E} \simeq \text{Sh}(\mathcal{O}(X))$ for some topological space X , then $\mathcal{E} \simeq \text{Et}(X)$.

Proof sketch

Our proof is substantially different from both Saadia's and Hamad's, who make use of Butz-Moerdijk's representation theorem for toposes with enough points. Instead, we proceed similarly to Makkai's original work, in two main steps.

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1. $\llbracket - \rrbracket: \mathcal{E} \longrightarrow \text{Et}(X)$ is **full on subobjects**: every subobject of $\pi_\varphi: \llbracket \varphi \rrbracket \longrightarrow X$ in $\text{Et}(X)$ is the restriction of π_φ to $\llbracket \psi \rrbracket \subseteq \llbracket \varphi \rrbracket$ for some subobject $\psi \rightharpoonup \varphi$ in $\mathcal{E}$.

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2. $\llbracket - \rrbracket: \mathcal{E} \longrightarrow \text{Et}(X)$ is **covering**: every étale space $p: Y \longrightarrow X$ is covered by an epimorphism $\alpha: \pi_\varphi \twoheadrightarrow p$ in $\text{Et}(X)$ for some object $\varphi \in \mathcal{E}$.

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Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $\llbracket - \rrbracket$.

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Two points of view

Concretely, (1) entails fully-faithfulness, while (2) entails essential surjectivity of $\llbracket - \rrbracket$. However, we can also interpret (1) as stating that $\llbracket - \rrbracket$ defines a hyperconnected geometric morphism, and (2) as stating that it defines a localic geometric morphism.

Ultraconvergence spaces as algebras

In joint work with Quentin Aristote, we pushed the inspiration from Barr's theorem even further.

Theorem (Aristote, T.)

The ultracompletion pseudomonad $\beta: \mathbf{CAT} \longrightarrow \mathbf{CAT}$ extends to a skew monad $\underline{\beta}$ on $\mathbf{PROF}$, and $\mathbf{UltSp} \cong \mathbf{LaxAlg}_{r,co}^{norm}(\underline{\beta})$.

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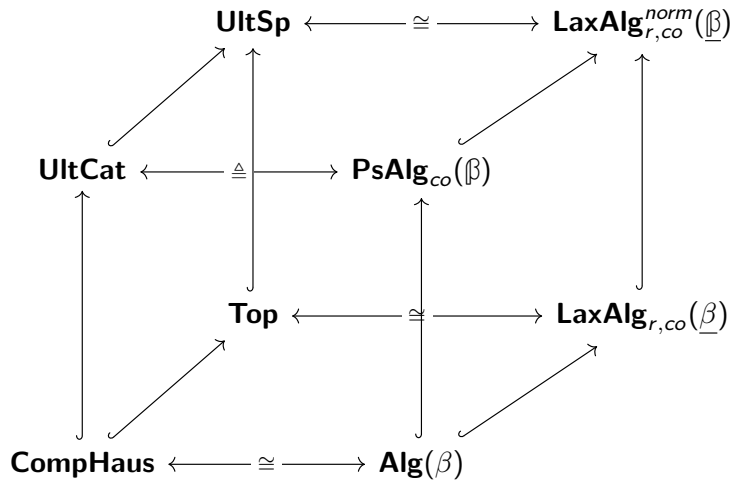
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This means that an ultraconvergence structure on a discrete category X can be equivalently specified by a profunctor $\Xi: \beta X \rightarrow X$ and two transformations

$$\begin{array}{ccc}
 X & \xlongequal{\quad} & X \\
 \eta_X \searrow & \Downarrow & \nearrow \Xi \\
 & \beta X &
 \end{array}
 \qquad
 \begin{array}{ccc}
 \beta^2 X & \xrightarrow{\beta \Xi} & \beta X \\
 \mu_X \downarrow & \swarrow & \downarrow \Xi \\
 \beta X & \xrightarrow{\Xi} & X
 \end{array}$$

satisfying usual coherence axioms.



Future work

- ▶ We can repeat the above story **filters** and **reduced products** in place of ultrafilters and ultraproducts: this yields a notion of **convergence spaces**, categorifying the relational algebras for the filter monad. How do these relate with ultraconvergence spaces? Can we obtain a more constructive version of the reconstruction theorem?

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Thank you!

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