

Tripeses and toposes via arrow algebras

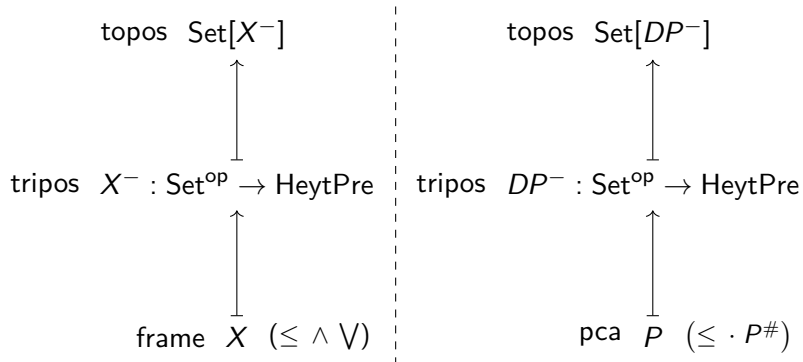
Umberto Tarantino

IRIF, Université Paris Cité

Categorical Logic and Higher Categories

18th December 2024

Localic toposes and realizability toposes



Question

Can we find a common framework to study both localic and realizability toposes from a concrete, “elementary” level?

Arrow algebras

Arrow algebras are a generalization of Alexandre Miquel's implicative algebras aimed to factor through the two constructions.

$$\begin{array}{ccc} \text{topos } \text{Set}[A^-] & & \\ \uparrow & & \\ \text{tripos } A^- : \text{Set}^{\text{op}} \rightarrow \text{HeytPre} & & \\ \uparrow & & \\ \text{arrow algebra } A (\leq \wedge \rightarrow S) & & \end{array}$$

Briët, van den Berg, *Arrow algebras*, 2023

Morphisms of arrow algebras

Morphisms between arrow algebras can be defined as functions “preserving” the implication \rightarrow and the subset S .

More precisely, a **morphism** $f : (A, \leq, \rightarrow, S_A) \rightarrow (B, \leq, \rightarrow, S_B)$ is a function $f : A \rightarrow B$ such that:

1. $f(S_A) \subseteq S_B$;
- 2.

$$\bigwedge_{a, a' \in A} f(a \rightarrow a') \rightarrow f(a) \rightarrow f(a') \in S_B;$$

3. for every $I \subseteq A \times A$,

$$\text{if } \bigwedge_{(a, a') \in I} a \rightarrow a' \in S_A \text{ then } \bigwedge_{(a, a') \in I} f(a) \rightarrow f(a') \in S_B.$$

Morphisms of arrow algebras

These morphisms specialize to:

- ▶ finite-meets preserving maps of frames;
- ▶ partial applicative morphisms of pcas,

and correspond exactly to:

- ▶ cartesian transformations of the induced triposes.

$$\begin{array}{ccc} \text{ArrAlg} & & \text{Tripos}_{\text{cart}} \\ A \xrightarrow{f} B & \simeq & A^- \xrightarrow{f \circ -} B^- \end{array}$$

T., *A category of arrow algebras for modified realizability*, 2024

Morphisms of arrow algebras

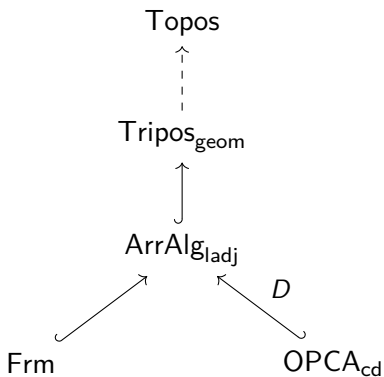
In particular, morphisms which are **left adjoints** in the preorder-enriched category ArrAlg specialize to:

- ▶ **homomorphisms** of frames;
 - ▶ **computationally dense partial applicative morphisms** of pcas,
- and correspond exactly to:
- ▶ **geometric transformations** of the induced triposes.

$$\begin{array}{ccc} & \text{ArrAlg}_{\text{gladj}} & \\ & \begin{array}{ccc} & f & \\ A & \curvearrowright & B \\ & \perp & \\ & h & \end{array} & \simeq \\ & & \begin{array}{ccc} & \text{Tripos}_{\text{geom}} & \\ & \begin{array}{ccc} & f \circ - & \\ A^- & \curvearrowright & B^- \\ & \perp & \\ & h \circ - & \end{array} & \end{array} \end{array}$$

T., *A category of arrow algebras for modified realizability*, 2024

2-categorically, we have the following picture:



Subtoposes and nuclei

As an example of an application, we can characterize **subtoposes** in terms of **nuclei** on the underlying arrow algebra, generalizing what happens for locales.

Proposition

Let A be an arrow algebra. Then, every subtopos of $\text{Set}[A^-]$ is induced by a geometric transformation of triposes

$$\begin{array}{ccc} & \text{id}_A \circ - & \\ & \curvearrowright & \curvearrowleft \\ A^- & & A_j^- \\ & \curvearrowleft & \curvearrowright \\ & j \circ - & \\ & \perp & \end{array}$$

for some nucleus j on A .

Thank you!