Triposes and toposes via arrow algebras

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Localic toposes and realizability toposes

$$
\begin{tikzcd}
\text{topos} & \mathsf{Set}[X^-] \\
\text{tripos} & X^- : \mathsf{Set}^{\mathsf{op}} \to \mathsf{HeytPre} \arrow[u, mapsto] \\
\text{frame} & X \quad (\leq \ \wedge \ \bigvee) \arrow[u, mapsto]
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{topos} & \mathsf{Set}[DP^-] \\
\text{tripos} & DP^- : \mathsf{Set}^{\mathsf{op}} \to \mathsf{HeytPre} \arrow[u, mapsto] \\
\text{pca} & P \quad (\leq \ \cdot \ P^{\#}) \arrow[u, mapsto]
\end{tikzcd}
$$

Question

Can we find a common framework to study both localic and realizability toposes from a concrete, "elementary" level?

Arrow algebras

Arrow algebras are a generalization of Alexandre Miquel's implicative algebras aimed to factor through the two constructions.

$$
\begin{tikzcd}
\text{topos} \quad \mathrm{Set}[A^-] \\
\text{tripos} \quad A^- : \mathrm{Set}^{\mathrm{op}} \to \mathrm{HeytPre} \arrow[u, mapsto] \\
\text{arrow algebra} \quad A \quad (\leq \ \bigwedge \ \to \ S) \arrow[u, mapsto]
\end{tikzcd}
$$

Briët, van den Berg, *Arrow algebras*, 2023

Morphisms of arrow algebras

Morphisms between arrow algebras can be defined as functions "preserving" the implication $\to$ and the subset S.

More precisely, a morphism $f : (A, \leq, \to, S_A) \to (B, \leq, \to, S_B)$ is a function $f : A \to B$ such that:

1. $f(S_A) \subseteq S_B$;
2. $$\bigwedge_{a,a' \in A} f(a \to a') \to f(a) \to f(a') \in S_B;$$
3. for every $I \subseteq A \times A$,
$$\text{if} \bigwedge_{(a,a') \in I} a \to a' \in S_A \text{ then } \bigwedge_{(a,a') \in I} f(a) \to f(a') \in S_B.$$

T., *A category of arrow algebras for modified realizability*, 2024

Morphisms of arrow algebras

These morphisms specialize to:

- finite-meets preserving maps of frames;
- partial applicative morphisms of pcas,

and correspond exactly to:

- cartesian transformations of the induced triposes.

$$\mathsf{ArrAlg} \qquad\qquad \mathsf{Tripos}_{\mathsf{cart}}$$

$$A \xrightarrow{\ f\ } B \qquad \simeq \qquad A^{-} \xrightarrow{\ f \circ -\ } B^{-}$$

T., *A category of arrow algebras for modified realizability*, 2024

Morphisms of arrow algebras

In particular, morphisms which are left adjoints in the preorder-enriched category ArrAlg specialize to:

- homomorphisms of frames;
- computationally dense partial applicative morphisms of pcas,

and correspond exactly to:

- geometric transformations of the induced triposes.

$$\mathrm{ArrAlg}_{\mathrm{ladj}} \qquad\qquad\qquad \mathrm{Tripos}_{\mathrm{geom}}$$

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "h"] \end{tikzcd} \quad \simeq \quad \begin{tikzcd} A^{-} \arrow[r, bend left, "f \circ -"] \arrow[r, phantom, "\perp"] & B^{-} \arrow[l, bend left, "h \circ -"] \end{tikzcd}$$

T., *A category of arrow algebras for modified realizability*, 2024

2-categorically, we have the following picture:

$$
\begin{tikzcd}
 & \mathsf{Topos} & \\
 & \mathsf{Tripos}_{\mathsf{geom}} \arrow[u, dashed] & \\
 & \mathsf{ArrAlg}_{\mathsf{ladj}} \arrow[u, hook] & \\
\mathsf{Frm} \arrow[ur, hook] & & \mathsf{OPCA}_{\mathsf{cd}} \arrow[ul, hook', "D"']
\end{tikzcd}
$$

Subtoposes and nuclei

As an example of an application, we can characterize subtoposes in terms of nuclei on the underlying arrow algebra, generalizing what happens for locales.

Proposition

Let A be an arrow algebra. Then, every subtopos of $\mathsf{Set}[A^-]$ is induced by a geometric transformation of triposes

$$\begin{tikzcd} A^- \arrow[r, bend left, "\mathrm{id}_A \circ -"] \arrow[r, phantom, "\perp"] & A_j^- \arrow[l, bend left, "j \circ -"] \end{tikzcd}$$

for some nucleus j on A.

Thank you!