A gentle introduction to categorical realizability

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Outline

1. Introduction
2. Towards a categorical model of realizability
3. Logic and computability in the effective topos
4. Conclusion

Introduction

History and motivation

Realizability was invented by Stephen Coole Kleene in 1945 in the attempt to make the computational content of constructive proofs explicit.
Concretely, realizability was the result of trying to find a connection between intuitionistic number theory and the theory of partial recursive functions.

Connectives and quantifiers, constructively

1. $\exists y\,(\,y + 3 = 7\,)$
2. $\forall x\,\exists y\,(\,x \geq 3 \rightarrow y + 3 = x\,)$
3. $\forall x\,(\exists y\,(2 \cdot y = x) \vee \neg\exists y\,(2 \cdot y = x)\,)$

History and motivation

"Intuitionistic number theory" is now known as Heyting arithmetic (**HA**), the intuitionistic counterpart of Peano arithmetic (**PA**).

Every **HA**-theorem is also a **PA**-theorem, but not the converse; we will see a counterexample later.

However, the two theories are equiconsistent.

Theorem (Gödel, 1933)

If **HA** *is consistent, then so is* **PA**.

The key idea

Notation

- Assume fixed a recursive bijection $\langle -, - \rangle : \mathbb{N}^2 \longrightarrow \mathbb{N}$.
- Assume fixed an enumeration of partial recursive functions $\mathbb{N} \rightharpoonup \mathbb{N}$ (equivalently, Turing machines), where $\{ e \}$ is the e-th function.
- Given a partial recursive function $f : \mathbb{N} \rightharpoonup \mathbb{N}$, we write $f(m)\downarrow$ to mean that f is defined on input $m \in \mathbb{N}$.

The key idea

Kleene's realizability associates to every sentence φ in the language of arithmetic a set of natural numbers, called its realizers, where $e \in \mathbb{N}$ realizes φ if:

- φ is atomic and it is true, and $e = 0$;
- $\varphi = \psi \wedge \chi$ and $e = \langle m, n \rangle$ where m realizes ψ and n realizes χ;
- $\varphi = \psi \vee \chi$ and either $e = \langle 0, m \rangle$ where m realizes ψ or $e = \langle 1, n \rangle$ where n realizes χ;
- $\varphi = \psi \to \chi$ and, for every realizer n of ψ, $\{ e \} (n)$ is defined and realizes χ;
- $\varphi = \exists x \psi(x)$ and $e = \langle m, n \rangle$ where m realizes $\psi(n)$;
- $\varphi = \forall x \psi(x)$ and, for all $n \in \mathbb{N}$, $\{ e \} (n)$ is defined and realizes $\psi(n)$.

The key idea

In particular, $\bot$ is never realized.

So, $\neg\varphi := \varphi \to \bot$ is realized if and only if φ does not have any realizer, in which case every $e \in \mathbb{N}$ realizes $\neg\varphi$.

- This means that, for every sentence φ,

  $$\varphi \vee \neg\varphi$$

  is always realized.

- Instead, if $\varphi(x)$ has a free variable x,

  $$\forall x\,(\,\varphi(x) \vee \neg\varphi(x)\,)$$

  is realized only if there exists a recursive function that tells, for each $n \in \mathbb{N}$, which of $\varphi(n)$ and $\neg\varphi(n)$ is realized.

Soundness of the realizability interpretation

Theorem

If a first-order sentence is provable in **HA***, then it is realized.*

The converse is false

Let $T(x, y, t)$ be the predicate saying that the x-th Turing machine halts on input y in less than t steps. A realizer of

$$\varphi := \forall x \, \forall y \, (\, \exists t \; T(x, y, t) \vee \neg \exists t \; T(x, y, t) \,)$$

would yield a recursive solution of the halting problem, which is impossible. So:

- φ is provable in **PA**, but not realized nor provable in **HA**;
- $\neg\varphi$ is realized, but not provable in **HA** nor in **PA**.

Towards a categorical model of realizability

$\mathcal{P}\mathbb{N}$-valued predicates

Abstracting the previous discussion, we define a $\mathcal{P}\mathbb{N}$-valued predicate on a set X as a function $X \longrightarrow \mathcal{P}\mathbb{N}$.

Predicates on X can be preordered by letting $\varphi \vdash_X \psi$ if

$$\exists e \in \mathbb{N}\ \forall x \in X\ \forall m \in \varphi(x) : \{\, e \,\}\,(m) \downarrow \wedge \{\, e \,\}\,(m) \in \psi(x)$$

To model logical connectives, we can define:

- $\top(x) = \mathbb{N}$ and $\bot(x) = \emptyset$
- $(\varphi \wedge \psi)(x) = \{\, \langle m, n \rangle \mid m \in \varphi(x) \text{ and } n \in \psi(x) \,\}$
- $(\varphi \vee \psi)(x) = \{\, \langle 0, m \rangle \mid m \in \varphi(x) \,\} \cup \{\, \langle 1, n \rangle \mid n \in \psi(x) \,\}$
- $(\varphi \to \psi)(x) = \{\, e \mid \forall m \in \varphi(x) : \{\, e \,\}\,(m) \downarrow \wedge \{\, e \,\}\,(m) \in \psi(x) \,\}$

Substitution and quantification

If $f : X \longrightarrow Y$ is any function, then the map

$$f^* : \mathcal{P}\,\mathbb{N}^Y \longrightarrow \mathcal{P}\,\mathbb{N}^X \qquad f^*(\varphi) := \varphi \circ f$$

preserves the order and the connectives.

Moreover, f^* has both a left and a right adjoint, $\exists_f$ and $\forall_f$, defined for $\varphi : X \longrightarrow \mathcal{P}\,\mathbb{N}$ by

$$(\exists_f \varphi)(y) := \bigcup_{x : f(x) = y} \varphi(x) \qquad (\forall_f \varphi)(y) := \bigcap_{x : f(x) = y} (\mathbb{N} \to \varphi(x))$$

which behave nicely with respect to substitution.

The association $X \mapsto (\mathcal{P}\,\mathbb{N}^X, \vdash_X)$ is known as the effective tripos.

The effective topos

The effective tripos gives rise to the effective topos Eff.

A topos is a category with enough categorical structure so that we can interpret essentially all mathematics inside it.

Example

The category Set of sets and functions is a topos, where standard mathematics takes place.

Internal vs external

The meaning of a sentence inside a topos can be externalized to a statement about the standard topos.

Realizability inside Eff

Eff admits a natural numbers object, an object which behaves like the set $\mathbb{N}$. In particular, it is a model of the theory of arithmetic inside Eff, so we denote it with $\mathbb{N}$ as well.

Theorem (Hyland, 1982)

In Eff*, a first-order sentence in the language of arithmetic is true in* $\mathbb{N}$ *if and only if it is realized.*

Corollary

In Eff*,* $\mathbb{N}$ *is a model of* **HA** *but not of* **PA***.*

Proof. $\mathbb{N}$ is a model of **HA** by soundness, but it doesn't validate the **PA**-theorem $\forall x \, \forall y \, (\exists t \, T(x, y, t) \vee \neg \exists t \, T(x, y, t))$. □

Logic and computability in the effective topos

Church's thesis

Eff admits power objects, and in particular the object $\mathbb{N}^{\mathbb{N}}$ of functions $\mathbb{N} \longrightarrow \mathbb{N}$. Church's thesis is the principle stating that every function $\mathbb{N} \longrightarrow \mathbb{N}$ is recursive:

$$\mathsf{CT} := \forall f \in \mathbb{N}^{\mathbb{N}}\ \exists e \in \mathbb{N}\ \forall m \in \mathbb{N} : \{\, e \,\}(m)\downarrow \wedge \{\, e \,\}(m) = f(m)$$

- CT is false in Set.

The *halting function* $h : \mathbb{N} \longrightarrow \mathbb{N}$ defined by

$$h(\langle e, m \rangle) := \begin{cases} 1 & \text{if } \{\, e \,\}(m)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

is famously not recursive.

Church's thesis

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- CT is trivially true in Eff.

In Eff, CT externally means that there exists a Turing machine M which, given a Turing machine F computing a function $f : \mathbb{N} \longrightarrow \mathbb{N}$, outputs a Turing machine computing f itself. Such an M trivially exists: it just echoes the input back.

Church's thesis

Why do the counterexamples in Set not work in Eff?

In particular, what happens to the halting function in Eff?

$$h(\langle e, m \rangle) = \begin{cases} 1 & \text{if } \{\, e \,\}\,(m)\downarrow \\ 0 & \text{otherwise} \end{cases}$$

The problem is that

$$\operatorname{dom} h = \{\, \langle e, m \rangle \in \mathbb{N} \mid \{\, e \,\}\,(m)\downarrow \vee \neg\{\, e \,\}\,(m)\downarrow \,\}$$

but we have already seen that

$$\forall e \in \mathbb{N}\ \forall m \in \mathbb{N}\ (\,\{\, e \,\}\,(m)\downarrow \vee \neg\{\, e \,\}\,(m)\downarrow\,)$$

is not true in Eff, so h is not provably total!

Constant zero functions

Consider instead the statement that every function $\mathbb{N} \longrightarrow \mathbb{N}$ is constantly zero or not:

$$\mathsf{Z} :\equiv \forall f \in \mathbb{N}^{\mathbb{N}} \left((\forall m \in \mathbb{N}\ f(m) = 0) \vee \neg (\forall m \in \mathbb{N}\ f(m) = 0) \right)$$

- Z is trivially true in Set.

In Set, the validity of Z is just an instance of the law of excluded middle.

Constant zero functions

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- Z is false in Eff.

In Eff, Z externally means that there exists a Turing machine M which, given a Turing machine F computing a function $f : \mathbb{N} \longrightarrow \mathbb{N}$, tells whether f is the constant zero function or not. Such an M does not exist: intuitively, the problem is that it would take *infinitely long* to be sure that f is the constant zero function.

Categoricity of Heyting arithmetic

PA is not categorical in Set, hence neither is **HA**.

The situation is quite different in Eff.

Theorem (van den Berg and van Oosten, 2014)

HA *is categorical in* Eff, *with* $\mathbb{N}$ *as the unique model.*

Corollary

PA *does not have any model in* Eff.

Corollary

Gödel's completeness theorem fails in Eff.

Conclusion

Variations on the theme

- We can build a topos based on infinite Turing machines (Hamkins and Lewis, 2000), where:
    i. any function $\mathbb{N} \longrightarrow \mathbb{N}$ is the constant zero function or not;
    ii. there are no surjections $\mathbb{N} \longrightarrow \mathbb{R}$, but there is an injection $\mathbb{R} \longrightarrow \mathbb{N}$ (Bauer, 2014).
- More generally, we can build toposes out of variants of Kleene's realizability, such as:
    i. a topos for function realizability, inside which every metric space is separable (Bauer and Swan, 2018);
    ii. toposes for extensional realizability, modified realizability, Lifschitz realizability...

Thank you!

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